Nonlinear Diffraction of Modulated Waves by a Thin Wedge

by

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Introduction

Herein we consider nonlinear waves incident to a long wedge. Under the assumptions of potential flow and forward diffraction, Yue and Mei (1980) obtained the cubic Schrödinger equation to describe the evolution of diffracted steady wave trains. The wave field shows the generation of stem waves along the wall which is closely related with the Mach reflection of shock waves. Miles (1977) clarified that the basic mechanism of the Mach reflection is the long-scale evolution of nonlinear waves. On the other hand, it is recalled that nonlinear wave trains are likely to be unstable, when they are subject to any small side-band disturbances (Benjamin & Feir,1967). Therefore it is natural to raise a question about the stability of the nonlinearly-diffracted wave field. As a first step towards this problem, it is aimed in this abstract to study the nonlinear diffraction of basically stably-modulated wave trains.

Formulation

Let us consider modulated waves with typical amplitude A_o incident to a long wedge stationary in deep water, of which the angle is 2α. The frequency and wave number of carrier waves are denoted by ω and k, respectively, and its modulation frequency by Ω as depicted in Fig.1. It is to note that the temporal and spatial variables involved in the problem are vastly different. We may classify them into three groups ; the spatial and temporal scales associated with carrier waves (x,t), those associated with modulation (K^{-1}, Ω^{-1}) and the length scale of the wedge (L). The orders of relative magnitude of these variables may be expressed in terms of the wave steepness, ε, like O(1), $O(\varepsilon^{-1})$, $O(\varepsilon^{-2})$, in this order, where ε is assumed small. To deal with these variables of different scales, it is relevant to employ multiple-scale expansion techniques. In accordance with the scheme, the velocity potential and the free surface are first expanded asymptotically with respect to ε and then further expanded into

Fourier series with respect to time t. It is well known that the first-order solution is trivial and the second-order solution claims the conservation of wave action. The evolution of wave amplitude is obtained at the third order as below :

$$\partial A/\partial X_2 + (i/4)(\partial^2 A/\partial X_1^2 - 2\partial^2 A/\partial Y^2) + i\delta^2|A|^2 A = 0. \qquad (1)$$

Here nondimensional variables are defined by

$$A = A/Ao,\ X_1 = kx_1,\ X_2 = kx_2,\ Y = ky_1,\ T = \omega t_1. \qquad (2)$$

and (X_1X_2) are long-scale longitudinal length variables and δ stands for the thinness of the wedge. This is known to be the Zakharov equation. The general form of the solution contains an integral constant which physically represents a current-like flow. In the above equation, this term has been simply discarded, because the current is small in deep water.

If it can be assumed that the diffracted waves as well as the incident waves are sinusoidally modulated, we finally obtain the cubic Schrödinger equation.

$$\partial A/\partial X_2 - (i/2)\partial^2 A/\partial Y^2 + i(-\mu^2 + \delta^2|A|^2) A = 0, \qquad (3)$$

where μ represents the dispersion ($\varepsilon\mu$ = K/2k).

The first term means the evolution of the wave amplitude with respect to the longitudinal distance, X_2, and the second one corresponds to its lateral diffusion, while the third term reflects both the dispersion and the nonlinearity. If the incident waves are uniform, *i.e.* $\mu = 0$, it reduces to the usual cubic Schrödinger equation Yue and Mei obtained. The above equation may be interpreted as the equation of motion for an oscillator of two-degrees of freedom with a nonlinear spring.

Numerical Result

For computations, the Crank-Nicholson algorithm has been utilized for X derivatives and the centered difference is taken for Y derivatives. To limit the dimension of the computation domian, a boundary condition is imposed at the numerical boundary, where $Y \gg 1$.

Computations are made for two wedges, whose half angles are 17.55^0(ε^2 = 0.1) and 24.09^0 (ε^2 = 0.2). Three wave steepnesses are considered, *i.e.*

kAo = 0.05, 0.1, 0.3. The corresponding values of δ are 0.1582, 0.3163, 0.9489 for α = 17.55^0 (ε^2 = 0.1) and 0.1118, 0.2236, 0.6708 for α = 24.09^0 (ε^2 = 0.2). Two modulations are taken, *i.e.* K/2k = 0.1 and 0.3. According to the result of Longuet-Higgins (1978), these modulations are stable except the cases of K/2k = 0.1 for kAo = 0.1 and 0.3. The nondimensional length of one modulation is $X_1 = \pi / 2\mu$ and it takes the nondimensional time of To = π / μ for carrier waves to propagate over this distance.

Fig.2 shows the snapshots of wave field for kAo = 0.3 and K/2k = 0.1, in which the ordinate represents the lateral coordinate Y. Attention should be paid to the different scales of the lateral length Y(= $k\varepsilon y$) from that of the horizontal length X_2(= $k\varepsilon^2 x$). The numbers denote the dimensionless wave amplitude. The overall features for two different wedge angles are similar each other. Stem waves are observed near the wedge and its width increases almost linearly downstream. The stem angle is measured to be 7.2^0 for the narrower wedge and 7.0^0 for the wider wedge. Although the difference is tiny small, its trend is in the direction to support the experiments.

All other numerical results sofar coincides with the experimental findings (Wiegel,1964) and indicate that the nonlinearity in the Schrödinger equation affects the wave evolution more strongly than the dispersion.

References

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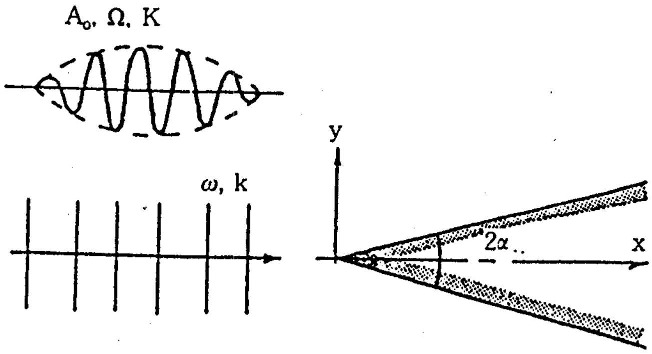
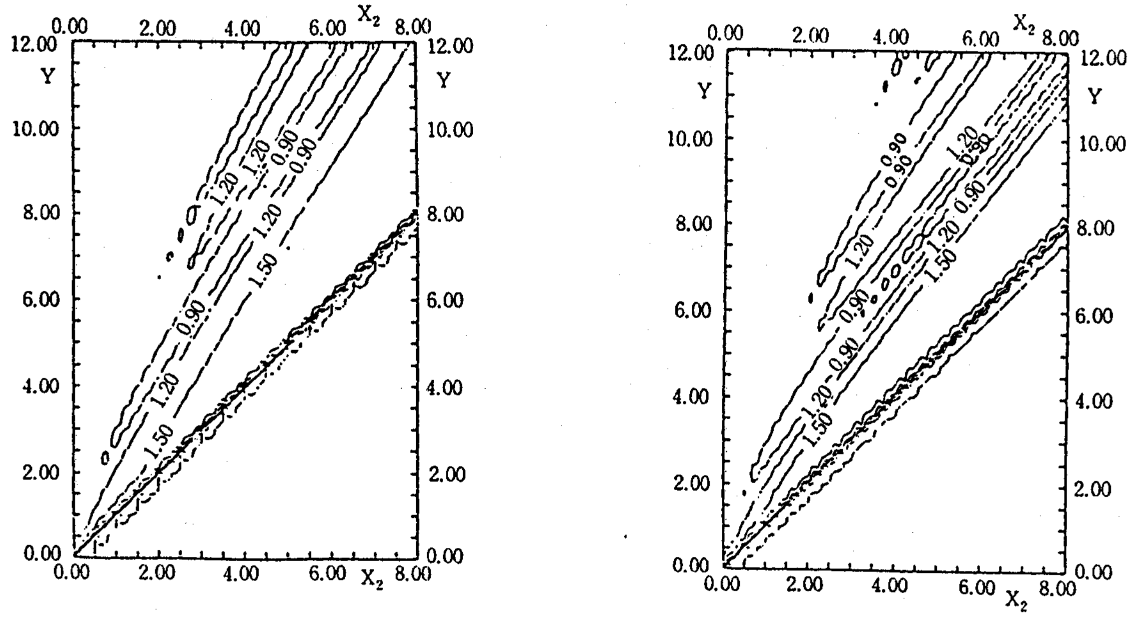


Fig.1 Definition Sketch



(a) ε^2 = 0.1

(b) ε^2 = 0.2

Fig.2 Contour of Diffracted Waves along the Wedge
(kA_o = 0.3 and μ = 0.1)

DISCUSSION

Korobkin A.: 1) Equations of gas dynamics are of hyperbolic type, but the nonlinear Schrödinger equation seems to be of parabolic type. Perhaps this is a reason why the Mach stem in your calculations was unstable.

2) I do not know another numerical scheme for NSE except Crank-Nicholson' one. What is your opinion, why this scheme is so popular? Is it the best one?

Choi Y.R. & Choi H.S. : The cubic Schrödinger equation is basically of hyperbolic type, which becomes clear if we consider the real and imaginary parts of the equation under the condition of $\frac{\alpha \bar{A}}{\alpha T_2} = 0$. The possible instability may come from the long-scale evolution of diffracted waves, which might has an analogy with side-band stability. The parabolic approximation is meant in our study that the diffracted waves propagate mostly in the direction of incident waves.

To your second question, we are not in a position to judge whether there is any better scheme than the Crank-Nicholson scheme we have used. The reason why we have used this scheme is that it is relatively stable and widely used for numerical computation of the cubic Schrödinger equation.

Yue D.K.P.: 1) It is quite remarkable that the effect of modulation is so small. Could you offer a simple explanation or physical argument that this should be so?

2) Given that the modulation does not appreciably affect the diffraction by the wedge, the question of instability can presumbly be addressed by following the evolution of (longitudinal and lateral) disturbances imposed on the steady(uniform incident wave) diffracted solution. Would you comment on this?

Choi Y.R. & Choi H.S.: The last term of the cubic Schrödinger equation we derived may be manipulated as a nonlinear spring with a potential well, i.e. the modulation is linear and the nonlinearity is cubic to the wave amplitude. The cubic nonlinear term dominates when the amplitude is relatively large as in this case. We conjecture that is a reason why the nonlinearity is more important than the modulation in our numerical results.

To your second question we fully agree to you. In fact we expected the diffraction by a wedge, in particular its oblique component as an agency which might trigger any instability of the wave field.