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ASYMPTOTIC EXPANSION OF THE CAUCHY-POISSON PROBLEM IN A FLUID OF FINITE DEPTH

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INTRODUCTION

The potential on the free surface due to a concentrated impulsive disturbance, acting at the origin at the initial time $t = 0$, can be expressed in the general form

$$\phi(x,t) = \int_0^\infty \begin{pmatrix} \frac{1}{\omega}\sin\omega t \\ \cos\omega t \end{pmatrix} \begin{pmatrix} \cos kx \\ kJ_0(kx) \end{pmatrix} dk. \qquad (1)$$

Here $\omega(k) = \sqrt{k\tanh k}$ is the dispersion relation for plane waves with wavenumber k and frequency ω. Nondimensional units are used with the fluid depth and gravity set equal to one. In two-dimensional problems x is the horizontal coordinate and the function $\cos kx$ is applicable; in the three-dimensional (axisymmetric) case x denotes the radial coordinate and the Bessel function $J_0(kx)$ applies. Two complementary solutions are considered: if the 'initial elevation' $\phi_t(x,t)$ is nonzero, the time-dependent function is $\frac{1}{\omega}\sin\omega t$; conversely, if the 'initial potential' $\phi(x,0)$ is prescribed the corresponding function in (1) is $\cos\omega t$. Similar integrals represent the free-surface effect for a submerged transient source, and hence the Green function required to solve boundary-value problems in the time domain (cf. Wehausen & Laitone, 1960, §13). Thus the integrals in (1) are frequently encountered. Appropriate asymptotic expansions for large time are required to give a more intuitive understanding, and to facilitate numerical computations.

In the infinite-depth case, where $\omega(k) = \sqrt{k}$, relatively simple asymptotic approximations are derived by Lamb (1932). For $t >> 1$, outgoing plane waves exist with local group velocity $v_g = x/t$. In finite depth the situation is more complicated, since the nondimensional group velocity cannot exceed the limit $v_g = 1$. Thus a 'front' at $x \sim t$ separates the waves behind from an evanescent region out ahead. Behind the front an asymptotic expansion can be derived analogous to the infinite-depth case, and the method of steepest descents can be used to describe the exponentially small disturbance ahead of the front. Both approximations fail near the front.

Several authors have derived approximations which are valid locally near the front, in terms of Airy functions; the first such work, by Kajiura (1963), was applied to Tsunami propagation. However these local approximations are limited in their domains of validity, and cannot always be matched up with the simpler approximations away from the front.

Our objective here is to present uniformly valid asymptotic expansions of (1) which span the three separate domains behind, at, and ahead of the front. Further details of the analysis, numerical examples, and additional references are included in Clarisse, Newman, & Ursell (1994), hereafter designated as 'CNU'.

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THE TWO-DIMENSIONAL CASE

The two complementary cases are defined separately. For the initial elevation,

$$\phi(x,t) = \int_0^\infty \sin\omega t \cos kx \frac{dk}{\omega} = \tfrac{1}{2}\int_0^\infty \left[\sin(\omega t - kx) + \sin(\omega t + kx)\right]\frac{dk}{\omega}, \tag{2}$$

and for the initial potential,

$$\psi(x,t) = \int_0^\infty \cos\omega t \cos kx dk = \tfrac{1}{2}\int_0^\infty \left[\cos(\omega t - kx) + \cos(\omega t + kx)\right]dk. \tag{3}$$

For $t = 0$, ϕ_t and ψ are equivalent to a delta function at the origin. In the analysis to follow it is important to note that $\omega(k) = \sqrt{k\tanh k} \sim k - \frac{1}{6}k^3 + ...$ is analytic except for branch cuts on the imaginary axis extending from $\pm i\pi/2$ to $\pm i\infty$.

For large positive values of x, t it is easy to show that the integral of the last term in (3) vanishes, and the contribution from the last term in (2) is $\pi/4$. The more interesting asymptotic contributions are from the terms with the oscillatory phase function $(\omega t - kx) \equiv t\Phi(k,a)$, where $a = x/t$ and $\Phi = \omega(k) - ka$. For $t >> 1$ the dominant contributions are from the vicinity of the points where $\Phi_k = 0$. For $a < 1$ two points of stationary phase $k = \pm k_0(a)$ are on the real axis, where $\omega' = v_g = a$. For $a > 1$ symmetric saddle points exist on the imaginary axis. These points coalesce at the origin when $a \to 1$. This is evident from the expansion

$$\Phi(k,a) = \sqrt{k\tanh k} - ka = (1-a)k - \tfrac{1}{6}k^3 + O(k^5), \quad \text{for } k << 1. \tag{4}$$

Following the method of Chester, Friedman & Ursell (1957), a new variable of integration $u = u(k;\epsilon)$ is introduced such that

$$\sqrt{k\tanh k} - ka = \epsilon u - \tfrac{1}{6}u^3. \tag{5}$$

Since the k-derivative of the left side of (5) vanishes when $k = \pm k_0(a)$, and the u-derivative of the right side vanishes when $u = \pm\sqrt{2\epsilon}$, these points must correspond and $\epsilon(a)$ is defined by

$$\Phi(k_0,a) = \sqrt{k_0\tanh k_0} - k_0 a = \frac{2^{3/2}}{3}\epsilon^{3/2}. \tag{6}$$

Note that $\epsilon \sim (1-a)$ for $a \sim 1$, and $\epsilon > 0$ behind the front. With ϵ so defined, (5) is regular and one-to-one near the origin and along the real axis.

The transformation (5) introduces the weight factor dk/du, which can be expanded in even powers of u with the leading term $\epsilon/(1-a)$. It follows that

$$\phi(x,t) \sim \tfrac{1}{2}\int_0^\infty \sin\left(t(\epsilon u - \tfrac{1}{6}u^3)\right)\frac{u}{\omega}\frac{dk}{du}\frac{du}{u} + \frac{\pi}{4} \sim \tfrac{1}{2}\int_0^\infty \sin\left(t(\epsilon u - \tfrac{1}{6}u^3)\right)\frac{du}{u} + \frac{\pi}{4}, \tag{7}$$

$$\psi(x,t) \sim \tfrac{1}{2}\int_0^\infty \cos\left(t(\epsilon u - \tfrac{1}{6}u^3)\right)\frac{dk}{du}du \sim \frac{\epsilon}{2(1-a)}\int_0^\infty \cos\left(t(\epsilon u - \tfrac{1}{6}u^3)\right)du. \tag{8}$$

The last integrals in (7-8) can be evaluated in terms of the Airy function Ai and its integral. Thus

$$\phi(x,t) \sim \frac{\pi}{2}\int_0^{2^{1/3}t^{2/3}\epsilon} \mathrm{Ai}(-z)dz + \frac{\pi}{6}, \tag{9}$$

$$\psi(x,t) \sim \frac{\pi\epsilon}{(1-a)}2^{-2/3}t^{-1/3}\mathrm{Ai}(-2^{1/3}t^{2/3}\epsilon). \tag{10}$$

Using the asymptotic expansions for the Airy function and its integral (Abramowitz & Stegun, 1964, eqs. 10.4.59-60 and 10.4.82-83), valid for large values of the argument, it follows that (9) and (10) are exponentially small ahead of the front; behind the front (9) is oscillatory about the mean value $\pi/2$ and (10) is oscillatory about zero.

THE THREE-DIMENSIONAL CASE

Only the solution for the initial elevation is considered here, defined by

$$\phi(x,t) = \int_0^\infty \frac{\sin\omega t}{\omega} J_0(kx)\, k\, dk = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin\omega t}{\omega}\, H_0^{(2)}(kx)\, k\, dk, \tag{11}$$

where $H_0^{(2)}$ denotes the Hankel function of the second kind and the path of integration is indented to pass below the origin. The Hankel function can be replaced by a suitable integral representation to give the double integral

$$\phi(x,t) = \frac{2^{-3/2}}{\pi}\int_{-\infty}^\infty kdk \int_{-\infty}^\infty d\sigma\, \frac{e^{i\omega t} - e^{-i\omega t}}{\omega}\, \frac{\exp\{-ikx(1+\sigma^2)\}}{(1+\frac{1}{2}\sigma^2)^{1/2}}. \tag{12}$$

The contribution from the term involving $e^{-i\omega t}$ is negligible.

Generalizing the procedure in (4-6), we change the variables of integration so that the argument of the exponential function is a polynomial. Thus, if new variables (u,v) are defined by (5) and by the relation $k\sigma^2 = uv^2$, the phase function is a cubic polynomial in (u,v):

$$\phi(x,t) \sim \frac{2^{-3/2}}{\pi}\int_{-\infty}^\infty du \int_{-\infty}^\infty dv\, G(u,v)\exp\{it(\epsilon u - \tfrac{1}{6}u^3 - uv^2 a)\}, \tag{13}$$

The slowly-varying function $G(u,v)$ includes the Jacobian of the transformation $(k,\sigma) \to (u,v)$. With the bilinear transformations $u = -2^{-1/3}(\xi+\eta)$, $v = -2^{-5/6}a^{-1/2}(\xi-\eta)$, (13) is replaced by

$$\phi(x,t) \sim \frac{2^{1/3}}{4\pi a^{1/2}}\int_{-\infty}^\infty d\xi \int_{-\infty}^\infty d\eta\, G^*(\xi,\eta)\exp(it\Psi), \tag{14}$$

where $G^*(\xi,\eta) = G(u,v)$ and $\Psi = \frac{1}{3}(\xi^3+\eta^3) - 2^{-1/3}\epsilon(\xi+\eta)$.

There are four saddle points $\nabla\Psi = 0$, symmetrically located at $\xi = \pm\xi_0$ and $\eta = \pm\eta_0$, where $\xi_0 = \eta_0 = 2^{-1/6}\epsilon^{1/2}$. Following Ursell (1980), the leading-order contribution to the asymptotic expansion of (14) follows from the approximation

$$G^*(\xi,\eta) \sim E_0 + A_0\xi + B_0\eta + C_0\xi\eta, \tag{15}$$

where the coefficients (E_0, A_0, B_0, C_0) are such that (15) is exact at the four saddle points. The contribution to (14) from each term in (15) is then evaluated from the integral

$$\int_{-\infty}^\infty d\xi \int_{-\infty}^\infty d\eta \exp\left(it(\tfrac{1}{3}\xi^3 + \tfrac{1}{3}\eta^3 + x\xi + y\eta)\right) = (2\pi)^2 t^{-2/3}\mathrm{Ai}(xt^{2/3})\mathrm{Ai}(yt^{2/3}) \tag{16}$$

and its partial derivatives, evaluated at $x = y = -2^{-1/3}\epsilon$. Since $G(u,v)$ is even in both variables,

$$\phi(x,t) \sim \frac{2^{1/3}\pi}{a^{1/2}t^{2/3}}(E_0\mathrm{Ai}^2 - C_0 t^{-2/3}\mathrm{Ai}'^2), \tag{17}$$

where the argument of the Airy function and its derivative is $-2^{-1/3}t^{2/3}\epsilon$. The coefficients E_0 and C_0, which depend on the values of G at the saddle points, are of order one. Near the front the term $E_0\mathrm{Ai}^2$ is dominant in (17). Away from the front, where the argument is large, the two terms in (17) are of the same order $(t^{-1/3})$, contributing more-or-less equally.

In CNU an analogous approximation is derived for the complementary function (1) with the time-dependant factor $\cos\omega t$; the final result involves only one term, proportional to the product AiAi'. The next higher-order terms in asymptotic expansions are derived for both cases. The complete asymptotic expansions involve the three functions Ai^2, AiAi', and Ai'^2, with coefficients which are proportional to G and its partial derivatives, evaluated at the saddle points.

DISCUSSION

Special attention has been given to deriving asymptotic approximations which are uniformly valid behind, near, and ahead of the front. If appropriate expansions are used for the Airy functions, the results are consistent with simpler approximations which apply away from the front, based on stationary phase or steepest descents. Earlier works including Kajiura (1963) and Newman (1992) derived local approximations near the front, using the cubic approximation in (4) for the phase function near the origin; this limits the domain of validity to $a \sim 1$ or $x \sim t$. The resulting asymptotic approximations are similar to (9-10), and to the first term in (17), but ϵ is replaced by $(1-a)$. Formally, the differences are small if and only if $|1-a| << t^{-2/5}$. The practical importance of this limitation is apparent from numerical results shown by Newman (1992).

This appears to be the first application of the methodology presented in Ursell (1980). Crucial steps are the use of an appropriate integral representation for the Hankel function, leading to the double integral (12), the change of variables which transforms the phase function in (13) to a cubic polynomial without approximation, and the second change of variables which yields the canonical form (14). Clarisse (1992) pursued the same objectives using singularity theory. Ursell (1993) discovered the appropriate transformation $(k,\sigma) \rightarrow (u,v)$ in a more *ad hoc* manner, motivated by the simpler two-dimensional problem.

The numerical accuracy of our uniform approximations is quite good. For the leading-order asymptotic approximations the maximum relative error for $t = 10$ is about 10^{-2}, improving to 10^{-3} for $t = 100$. For the two-term asymptotic expansions these are refined to 10^{-3} and 10^{-5}, respectively. The relative errors are insensitive to a throughout the range $0.1 < a < \infty$.

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