On the transient analysis of the wave maker

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Introduction

A time-domain finite element formulation is being developed for fully non-linear wave body interactions. As is well known, numerical analysis of such interactions raises a number of awkward problems. How best does one propagate the incident wave past the structure? How can the numerical domain be limited while still permitting the propagation of waves far downstream? How should the singularity at the waterline of a floating body be resolved? To provide the groundwork for dealing with such questions we are investigating the analytical solution for the wave maker. Based on a perturbation approach, we are generating numerical results which will be compared with those from the finite element formulation.

The linear analytical solution may be compared with that of Kennard (1949). His final form of solution differs from ours, and he did not give any numerical results. These were provided, however, by Dommermuth et al. (1988), based on an extension of Kennard's analysis for a finite length tank. Their results were obtained for a complex motion of the wave maker, designed to produce a very large wave (in reality a breaking wave) at a point far downstream. In the following our linear analysis is summarised, and some results given. These will be compared with results from the non-linear finite element formulation.

Analysis

It is useful, first, to write the formulation for the general case of a three dimensional body undergoing arbitrary motions at the free surface. We use a Cartesian coordinate system (x,y,z) located in the mean free surface, with z pointing upwards. Points in the fluid are denoted $\boldsymbol{p}$, and the underwater surface is S_B. We consider the transient problem represented by a velocity potential $\phi(\boldsymbol{p},t)$, satisfying the linear free surface condition, a condition on the seabed and a kinematic condition on the moving solid boundary. The initial conditions are taken as

$$\phi=0, \qquad \frac{\partial\phi}{\partial t}\Big|_{z=0} = 0, \qquad t=0 \tag{1}$$

The motion of the solid boundary S_B is taken as $v(t)f(p)$, so that the boundary condition on S_B is

$$\frac{\partial\phi}{\partial n}=v(t)\ \boldsymbol{f}.\boldsymbol{n} \tag{2}$$

where $\boldsymbol{n}$ is the normal at $\boldsymbol{p}$. The general solution is then

$$\phi(p,t)=\int_{-\infty}^{\infty} V(\omega)\varphi(p,\omega)e^{i\omega t}d\omega \tag{3}$$

where $V(\omega)$ is the Fourier transform of $v(t)$. $\varphi(p,\omega)$ may be subdivided as follows

$$\varphi(p,\omega)=\psi(p) + [\varphi(p,\omega)-\varphi(p,\infty)] \tag{4}$$

where ψ is used to denote the value of φ as $\omega\rightarrow\infty$. It is useful also to define

$$\chi(p,t)=\int_{-\infty}^{\infty} [\varphi(p,\omega)-\varphi(p,\infty)]e^{i\omega t}d\omega \tag{5}$$

Use of these in equation (2) leads to

$$\phi(p,t)=\psi(p)\ v(t)\ +\int_{-\infty}^{\infty}\chi(p,t-\tau)v(\tau)d\tau \tag{6}$$

The first term corresponds to the instantaneous effect of $v(t)$, and the second term is the memory effect.

On the basis that $v(t) = 0,\ t<0$, and that the system is causal, the lower and upper limits may be replaced by 0 and t respectively. On the free surface, ψ and χ satisfy the following conditions for $t > 0$.

$$\psi = 0 \tag{7}$$

$$\frac{\partial^2\chi}{\partial t^2}+g\frac{\partial\chi}{\partial z}=0 \tag{8}$$

The initial conditions on χ are

$$\chi(p,0) = 0 \tag{9}$$

$$\frac{\partial\chi}{\partial t}(p,0) = -g\frac{\partial\psi}{\partial z}\ ,\qquad z = 0 \tag{10}$$

On the body surface S_B

$$\frac{\partial\psi}{\partial n} = f.n,\qquad \frac{\partial\chi}{\partial n} = 0 \tag{11}$$

The wave maker

We now consider a wave maker in water of depth h, moving sinusoidally at frequency ω. For simplicity in the resulting expressions, the motions are such that

$$v(t) = \sin\omega t\ ,\qquad f.n = \frac{\cosh\ k_0(z+h)}{\cosh\ k_0h} = F(k_0z) \tag{12}$$

where $\omega^2 = k_og\ \tanh k_oh$. The solutions may be obtained as

$$\psi(x,z) = \sum_{n=1}^{\infty}C_n\cos a_n(z+h)e^{-a_nx} \tag{13}$$

where

$$C_n = -\frac{2}{a_nh}\int_{-h}^{0}F\ (k_0\xi)\cos a_n(\xi+h)d\xi,\qquad a_n=(n - \frac{1}{2})\frac{\pi}{h}$$

and

$$\chi(x,z,t) = \frac{2g}{\pi}\int_{0}^{\infty}\cos kxF(kz)\frac{\sin\beta t}{\beta}\int_{-h}^{0}F(k_0\xi)F(k\xi)d\xi dk \tag{14}$$

where $\beta^2 = kg \tanh kh$. This leads to the wave elevation $\zeta(x,t)$ which can be expressed in the form

$$\zeta(x,t) = A_0 \sin k_0 x \cos\omega t + \frac{2k_0 A_0}{\pi} \int_0^{\infty} \frac{\cos kx \cos\beta t - \cos k_0 x \cos\omega t}{k^2 - k_0^2} dk \qquad (15)$$

One finds that as $t \to \infty$ this provides the steady state solution, with amplitude $A_0 = \omega/(gk_0)$:

$$\zeta_s = A_0 \cos(k_0 x - \omega t) \qquad (16)$$

One also finds that as $x \to \infty$ the disturbance in the far field tends to zero as required.

Asymptotic analysis

Miles (1962) has used an asymptotic analysis to show that, away from the wavemaker, the amplitude ratio A/A_0 near the wave front may be expressed in terms of Fresnel sine and cosine integrals (The result is also given by Mei (1983)). Here A is the local wave amplitude prior to achievement of the steady state, so that $A(t)$ is the envelope process. This analysis is for deep water, and shows that at the wave front given by $x = c_g t$, where c_g is the group velocity, the local amplitude is half the steady state amplitude. The time interval between two successive wave crests coinciding with a peak in the envelope process is twice the wave period.

Results

Figure 1 shows results for a wave front corresponding to a wave of unit amplitude. The envelope predicted by Miles' theory is shown alongside the result based on equation 15, 10.4 seconds after starting the wave maker at a frequency of 6.48 rad/s. Figure 2 shows the corresponding comparison of the time history of wave elevation, at a fixed point 15m in front of the wave maker. Figure 3 shows the evolution of the front, each wave profile in the figure being separated from its neighbour by 1/8 of the wave period.

These and further results from the perturbation analysis will be used in the validation and interpretation of results from the finite element formulation (Wu and Eatock Taylor,1994).

References

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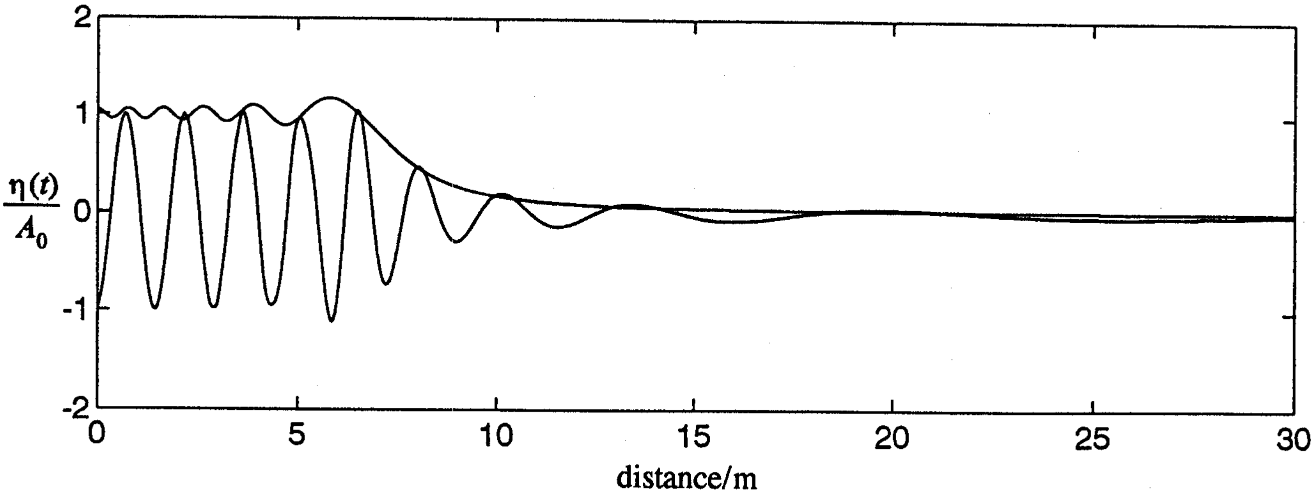

Figure 1 Wave elevation 10.4s after starting wavemaker compared with envelope from theory of Miles (1962)

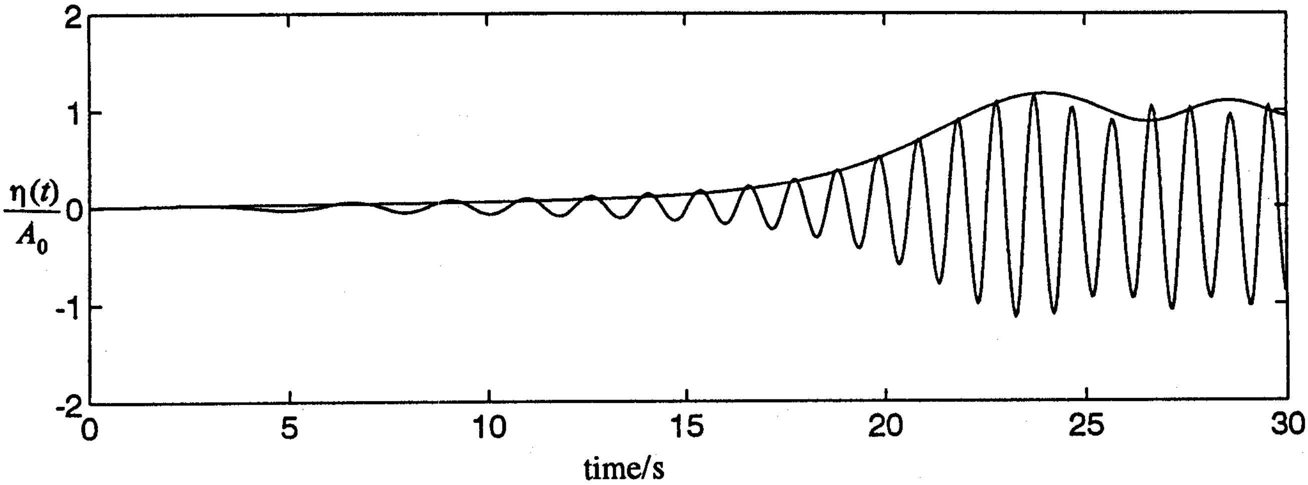

Figure 2 Time history of elevation 15m in front of wavemaker compared witn envelope from theory of Miles (1962)

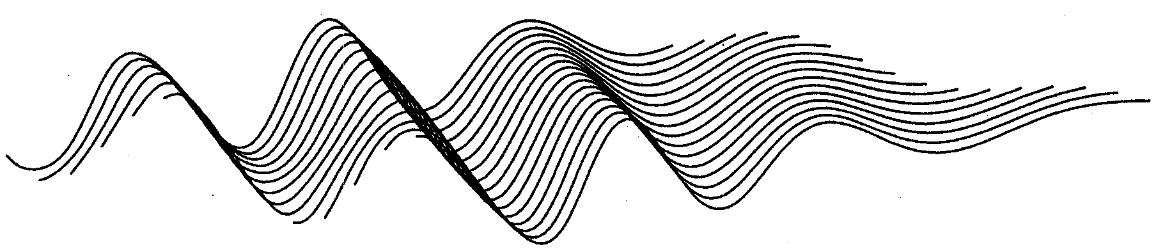

Figure 3 Evolution of the wave front

DISCUSSION

Clement A.: Did you compare the finite element approach to the boundary element method both applied to this problem, in term of computation burden (CPU consumption and data stroge volume), in the perspective of extension to 3D nonlinear problems?

Eatock Taylor R.: We have thought about this before embarking on this project, though I cannot give you explicit comparisons. Among reasons for our reinsisting the finite element approach are the following. 1) Concern about behavior at (and also near) the intersection of the body with the free surface. 2) The scope for relaxing the assumption of inviscid flow. 3) The possibility of applying novel adaptive mesh techniques and solution algorithms in the finite element formulation.

Tulin M.: 1) The front of the groups will inevitably be modified by non-linear effects, leading to a propagating wave groups there, governed by the cubic Schrödinger evolution equation. This has been well studied by Prof.Choi (Seoul Univ.) and presented at ISOPE '92 in San Francisco. He showed good agreement with experiments. 2) In our LONGTANK exact numerical simulations we eventually find the same effects, whereas early in the life of the wave train we find a large wave at the front, which always breaks before the waves behind. I hope these comments will be of use.

Eatock Taylor R.: We haven't yet produced results from our non-linear finite element method for very long wave trains, but we expect to do so. We shall bear your comments in mind.

Cao Y.: In our nonlinear calculations of waves generated by a wave maker starting from rest using an explicit time stepping scheme, at the first time instant, we usually have to solve a boundary value problem for the flow with the exact non-penetration condition on the wave maker and $\Phi = 0$ on the undisturbed free surface. This boundary value problem is not well defined and the solution to the BVP is singular at the intersection point of the wave maker and the free surface. This will cause the time stepping to stop if no "special" treatment is applied. The "special" treatments we use include 1) not to solve the BVP at the first time instant exactly so we do not get the infinite flow velocity at the intersection point; 2) the start-up error is scaled down by a very slow start-up of the wave maker or by using a very small time step size. With the use of a stable time stepping scheme, the start-up error would then not be likely to cause the simulation to break down. However, these treatments increase the computational cost.
I think your linear analytic solution for small time may be used for the first time instant, thus avoiding the start-up difficulty without using the "special" treatments. In your analytic solution, is the flow velocity at the intersection point finite for a general smooth start-up of the wave maker? What happens to the flow velocity at the intersection point if the wave maker has a sudden start-up?

Eatock Taylor R.: The order of the singularity depends on the movement of the wave-maker. For a sudden starup, similar to the impulse response in our analytical solution, the complex potential has a $z \ln z$ behaviour near the intersection. For a general smooth start-up, when the wave maker has small acceleration relative to gravity, the local behavior of the complex potential can be shown to be of the form $z^2 \ln z$. Regardless of any singularity in the vertical velocity at the intersection, there is no singularity in the wave elevation. In our finite element formulation, where we explicity impose both free surface and body surface conditions at the intersection point on the wavemaker, no special treatment is required. It is found that the wave elevation at the wavemaker obtained from the linearised analytical solution and from the non-linear numerical solution are in excellent agreement.