On step approximations for water wave problems

D V Evans
Department of Mathematics
University of Bristol
Bristol BS8 1TW,UK

December 22, 1993

Extended abstract

The problem of determining the scattering of a long-crested gravity wave over a bottom of arbitrary shape has received considerable attention in the literature. Despite the simplification afforded by linearization of the free surface condition, there appears to be only one explicit solution for a particular bottom shape. This solution, due to Roseau (1976) provides an explicit representation for the two-dimensional velocity potential of the field produced by a time-harmonic plane wave incident upon an underwater bottom topography having a smooth variation, in the direction of wave propagation only, from one constant depth to another. For a general bottom topography of this type, but where the depths at either side tend asymptotically to the same values, numerical methods are available, by utilizing the Green's function for a time harmonic wave source in finite depth. Application of Green's theorem provides a Fredholm integral equation for the unknown velocity potential over the non-constant part of the bottom profile whilst a separate integral representation provides the potential at an arbitrary point in the fluid region in terms of the potential on this bottom profile.

Of more interest, and more difficulty, is the problem of the bottom profile joining regions of *different* constant depths, since this corresponds to waves propagating over, for instance, a continental shelf of a given shape. The numerical technique described above is no longer available because of the difference in depths. This would need to be modified by using a Green's function appropriate to this case and again deriving an integral equation over the variable part of the bottom profile. Such a function has been developed (Evans 1972) for precisely this purpose, but its complicated nature makes the numerical procedure unwieldy.

The simplest problem of this type, the propagation of waves across a vertical step, was considered by Miles (1967) using eigenfunction expansions in each of the separate regions of constant depth. By invoking a variational approximation he was able to obtain accurate results in good agreement over a range of incident wavelengths, with the work of Newman (1965) who had considered the same problem when one of the depths was infinitely large.

Fitz-Gerald (1976) considers an arbitrary bottom profile in two dimensions and exploits complex variable techniques to map the two-dimensional fluid regions into an infinite strip at the expense of a more complicated free surface condition. This enables him to reduce the problem to the solution of an integro-differential equation from which he is able to prove uniqueness of the problem for general bottom profiles in the limits of wavelength, either large or small compared to the transition width joining the unequal constant depths.

More recently Devillard *et al* (1988) in extending the ideas of Anderson localization to water waves over random bottom profiles, have devised a transition matrix method for arbitrary bottom slopes to solve the full linear reflection and transmission problem. They make two approximations to achieve this end. First they assume that the bottom profile can be discretized into a series of horizontal shelves each of which has a horizontal length which is large compared to its height. This enables them to regard the local evanescent field generated at one shelf as negligible by the time it reaches the next. This wide-spacing approximation is effectively that used by Newman (1965) described above, and has been used by numerous authors in the water wave context to good effect. See for example Evans (1990) and Martin (1984). Their second approximation is to utilize Miles' (1967) variational approximation to relate the propagating waves on either side of the shelf through an appropriate 2×2 transition matrix. This enables them to obtain the cumulative effect of all the shelves on the reflection and transmission of an incident wave by multiplication of the transition matrices appropriate to each individual shelf. The idea has recently been taken up by O'Hare & Davies (1992) who have shown that the method works well both for rapidly varying bottom profiles and when the transition matrix is simplified still further by using a plane-wave approximation suggested by Miles (1967). They use the method to consider the reflection by a series of sinusoidal bottom profiles and they are able to reproduce accurately the high reflection due to Bragg resonance when the wavelength is twice the wavelength of the bottom profile. Their results suggest that the method may have wider validity than expected.

In the present paper we shall show how the idea of Devillard *et al* (1988) can be adapted to deal with bottom profiles which are neither smooth nor low, in the sense that the horizontal extend of the corresponding discretized shelves modelling the profile are large compared to the height of the shelves. The idea is first to map the fluid regions into an infinite strip as described by Fitz-Gerald (1976) thereby transferring the difficulty associated with the variable geometry to one involving a variable boundary condition on the free surface. The advantages of this are twofold. First, the effect of the mapping is to transform the equation of the bottom profile into a smoother surface condition. We now apply the discretization idea of Devillard *et al* (1988) to this variable free surface, anticipating greater accuracy because of the smoothing which has taken place. But there is a much more important advantage in discretizing the free surface condition rather than the bottom condition. Whereas Devillard *et al* need the solution over a shelf given by Miles, which is expressed only approximately either in terms of a plane-wave or a variational approximation, the discretization of the free surface condition requires the solution in a strip of constant width for the reflection and transmission of a wave incident from a semi-infinite region supporting waves of one wavenumber into a semi-infinite region supporting waves of a slightly different wavenumber. This problem has an explicit simple solution using the Wiener-Hopf technique which has been given by Weitz & Keller (1950) who were concerned with modelling the propagation of waves into a region containing small pieces of floating ice which had the effect of modifying the free surface condition. Equipped with this exact solution, we can, as in Devillard *et al* (1988) construct a series of transition matrices carrying information about the propagating waves across the region of varying surface condition. Here too we shall assume the effect of the local evanescent modes from one junction is negligible at the next.

In deriving the transition matrix use is made of general reciprocity conditions linking the left and right reflection and transmission coefficients so that the transition matrix reduces finally to the product of four separate 2×2 matrices; three rotation matrices involving 'angles' related to the relative size of the particular shelf, and the phases of a left and right reflection coefficient, and a matrix involving the modulus of a left reflection coefficient and the ratio of successive wavenumbers.

Some examples of the application of the method will be described. In particular the method will be applied to the explicit simple solution derived by Roseau (1976) when the bottom profile joins smoothly two regions of unequal depth. A comparison will be made between the Devillard approach and the discretized free-surface approach and it will be shown that in this case the latter is superior in approximating the exact answer for the modulus of the reflection coefficient. Two other examples, where solutions can be computed using alternative methods will also be used to validate the method. These are the scattering of waves by a single vertical step, and by a vertical barrier attached to the sea bed and extending part way to the free surface. In each case the transition matrix method proves to be simpler to implement compared to eigenfunction expansion methods, and also produces accurate results. The submerged barrier problem, which cannot be tackled by the method of Devillard *et al*, is a particularly severe test of the present method, yet good accuracy is achieved.

References

Devillard, P, Dunlop, F & Souillard, B 1988 Localization of gravity waves on a channel with random bottom. *J Fluid Mech* **186** 521-538.

Evans, D V 1972 The application of a new source potential to the problem of the transmission of water waves over a shelf of arbitrary profile. *Proc Camb Phil Soc* **71** 391-410.

Evans, D V 1990 The wide-spacing approximation applied to multiple scattering and sloshing problems. *J Fluid Mech* **210** 647-658.

Fitz-Gerald, G F 1976 The reflexion of plane gravity waves travelling in water of variable depth. *Phil Trans Roy Soc* **34** 49-89.

Martin, P A 1984 Multiple scattering of surface water waves and the null-field methods. *Proc 15th Symp Naval Hydrodynamics, Hamburg* pp119-132. National Academy Press.

Miles, J W 1967 Surface-wave scattering matrix for a shelf. *J Fluid Mech* **28** 755-767.

Newman, J N 1965 Propagation of water waves over an infinite step. *J Fluid Mech* **23** 399-415.

O'Hare, T J & Davies, A G 1992 A new model for surface wave propagation over undulating topography. *Coastal Engineering* **18** 251-266.

Roseau, M 1976 *Asymptotic Wave Theory.* North-Holland, Amsterdam.

Weitz, M & Keller, J B 1950 Reflection of water waves from floating ice in water of finite depth. *Comm Pure Appl Math* **3** 305-318.

DISCUSSION

Korobkin A.: 1) When the bottom slope is slight, the ray method can be used. It is of interest to compare your approach and the ray one. It can be expected that a slightly curved-bottom topography case is that for which the accuracy of your approach will be very high.

2) There is a paper of Kriesel who used conformal mapping to estimate the amplitudes of transmitted and reflected waves. It is interesting to compare your numerical results with his estimations.

Evans D.V.: For small bottom slopes, Devillard's method works well and we do not need to use the present method. It is most useful for steep bottom topographies when neither Devillard nor ray theory works since the reflection coefficient is relatively large.

Palm E.: In your method, the building stone is the step function. I understand that by mapping the fluid area into an infinite strip you reduce the problem to one which has only a discontinuity of the free surface. Hence by using the Wiever-Hopt technique the problem is essentially replaced by finding the transformation formula.

Evans D.V.: Yes, precisely, but whereas the step solution is only known approximately, the free-surface discontinuity problem has an exact solution. Also the conformal mapping has the effect of changing the bottom profile into a smoother free surface condition, so that a discretisation procedure is more accurate.

Liu Y.: Can the method be simply extended to the case of general three-dimensional bottom topography?

Evans D.V.: Unfortunately not since the method depends upon the use of conformal mappings which are not available for three-dimensional problems.