NON LINEAR HIGH-FROUDE-NUMBER SLENDER BODY THEORY

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1 Introduction

We consider the problem of calculating the wave resistance of a high-speed ship. The solution procedure relies on the method of matched asymptotic expansions. The ship is assumed slender and the Froude number $Fr = \frac{U}{\sqrt{gL}}$ (U is the speed of the ship, L its length) high. The perturbation parameter is taken to be $\epsilon = \frac{R}{L} \ll 1$ where R is the maximum transverse dimension of the ship. Our approach follows closely that of Ogilvie [3], but we use an asymptotic expansion rather than an asymptotic sequence and we explicitely write a Kutta condition at the stern. We are also able to solve the nonlinear inner problem numerically, using the Sindbad code [1] so that a full composite solution can be provided.

2 Inner solution

The ship is placed in a uniform stream with velocity $U \cdot \vec{e_x}$ $(U > 0)$. The z axis is vertical upward and the right handed coordinate system (x, y, z) is fixed with the ship. In addition, we use cylindrical coordinates r and θ given by $y = r\cos(\theta)$, $z = r\sin(\theta)$. We assume the fluid to be incompressible and the flow irrotational; we neglect surface tension and the pressure is constant along the free-surface. The total velocity potential can be written $\phi(x, y, z) = U\,x + \varphi(x, y, z)$. In order to perform the appropriate simplifications, the variables are first non-dimensionalized taking into account the length scales of the ship. We define:

$$\tilde{x} = \frac{x}{L}, \quad \hat{r} = \frac{r}{R}, \quad \hat{y} = \frac{y}{R}, \quad \hat{z} = \frac{z}{R}, \quad \hat{\eta} = \frac{\eta}{R}, \quad \hat{\varphi} = \frac{\varphi}{UL} \tag{1}$$

where $z = \eta(x, y)$ is the free surface elevation. The choice of L as length scale for the potential is arbitrary. The ship boundary is given by $\hat{r} = h(\tilde{x}, \theta)$ and the following asymptotics expansions are assumed:

$$\hat{\varphi}(\hat{x}, \hat{r}, \theta, \epsilon) = \hat{\mu}_1(\epsilon)\,\hat{\varphi}_1(\tilde{x}, \hat{r}, \theta) + o(\hat{\mu}_1), \quad \hat{\eta}(\tilde{x}, \hat{y}, \epsilon) = \hat{\nu}_1(\epsilon)\,\hat{\eta}_1(\tilde{x}, \hat{y}) + o(\hat{\nu}_1) \tag{2}$$

The first order perturbation potential satisfies two-dimensional Laplace equation in each plane $x = Cste$.

$$\frac{\partial^2 \hat{\varphi}_1}{\partial \hat{r}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{\varphi}_1}{\partial \hat{r}} + \frac{1}{\hat{r}^2}\frac{\partial^2 \hat{\varphi}_1}{\partial \theta^2} = 0 \tag{3}$$

Three-dimensionnal effects that appear far from the ship can not be taken into account and obviously the solution must be interpreted as an inner solution.
The body boundary condition gives:

$$\hat{\mu}_1\,[\frac{\partial \hat{\varphi}_1}{\partial \hat{r}} - \frac{1}{\hat{r}^2}\frac{\partial \hat{\varphi}_1}{\partial \theta} \cdot \frac{\partial h}{\partial \theta}] = \epsilon^2\,(1 + \hat{\mu}_1 \frac{\partial \hat{\varphi}_1}{\partial \tilde{x}}) \cdot \frac{\partial h}{\partial \tilde{x}}(\tilde{x}, h(\tilde{x}, \theta), \theta) \tag{4}$$

If $\hat{\mu}_1 \ll \epsilon^2$ this condition degenerates and an inner (inner) domain that takes into consideration the body boundary condition must be introduced. This corresponds to the jet domain defined by Tulin and Hsu [4]. Although their theory shows good agreement with experimental results, we chose the other possibility: $\hat{\mu}_1(\epsilon) = \epsilon^2$. This choice wich allows a jet solution to develop at the intersection of the free surface and the hull seems to be more general. It is expected this theory will able to treat a large range of Froude number.
Equation (4) reduces to:

$$[\frac{\partial \hat{\varphi}_1}{\partial \hat{r}} - \frac{1}{\hat{r}^2}\frac{\partial \hat{\varphi}_1}{\partial \theta} \cdot \frac{\partial h}{\partial \theta}] = \frac{\partial h}{\partial \tilde{x}}(\tilde{x}, h(\tilde{x},\theta), \theta) \tag{5}$$

The kinematic free surface condition has the form:

$$[\frac{\partial \hat{\eta}_1}{\partial \tilde{x}} + \frac{\partial \hat{\varphi}_1}{\partial \hat{y}} \cdot \frac{\partial \hat{\eta}_1}{\partial \hat{y}} - \frac{1}{\hat{\nu}_1}\frac{\partial \hat{\varphi}_1}{\partial \hat{z}} + \epsilon^2 \frac{\partial \hat{\varphi}_1}{\partial \tilde{x}} \cdot \frac{\partial \hat{\eta}_1}{\partial \tilde{x}}](\tilde{x}, \hat{y}, \hat{\nu}_1 \cdot \hat{\eta}_1) = 0 \tag{6}$$

The free surface-elevation is expected to be at most of the same order of magnitude as the maximum transverse dimension of the ship. If $\hat{\nu}_1 = o(1)$ then equation (6) reduces to the classical condition for small Froude number approximation $\frac{\partial \hat{\varphi}_1}{\partial \hat{z}}(\hat{x}, \hat{y}, 0) = 0$ wich is physically not acceptable in this problem . Choosing $\hat{\nu}_1 = 1$ leads to the richest form:

$$[\frac{\partial \hat{\eta}_1}{\partial \tilde{x}} + \frac{\partial \hat{\varphi}_1}{\partial \hat{y}} \cdot \frac{\partial \hat{\eta}_1}{\partial \hat{y}} - \frac{\partial \hat{\varphi}_1}{\partial \hat{z}}](\tilde{x}, \hat{y}, \hat{\eta}_1) = 0 \tag{7}$$

The dynamic free-surface condition becomes:

$$[\frac{\partial \hat{\varphi}_1}{\partial \tilde{x}} + \frac{1}{2}(\frac{\partial \hat{\varphi}_1}{\partial \hat{y}})^2 + \frac{1}{2}(\frac{\partial \hat{\varphi}_1}{\partial \hat{z}})^2 + \frac{1}{2}\epsilon^2(\frac{\partial \hat{\varphi}_1}{\partial \tilde{x}})^2](\tilde{x}, \hat{y}, \hat{\eta}_1) + \frac{\hat{\eta}_1}{\epsilon\, Fr^2} = 0 \tag{8}$$

The Froude number in this case depends on ϵ. In order to find a non-trivial solution for $\hat{\eta}_1$, we must impose $Fr \geq \frac{1}{\sqrt{\epsilon}}$. This is consistent with a high Froude number theory. If the Froude number is higher than $\frac{1}{\sqrt{\epsilon}}$, gravity effects can be neglected. This again corresponds to the high-speed theory of Tulin and Hsu [4] which is the infinite Froude number asymptotic solution. In that case, the last term of equation (8) is small an should not affect the solution. We define: $\hat{Fr} = Fr\,\sqrt{\epsilon}$, the resulting first order dynamic free-surface condition is:

$$[\frac{\partial \hat{\varphi}_1}{\partial \tilde{x}} + \frac{1}{2}\Big((\frac{\partial \hat{\varphi}_1}{\partial \hat{y}})^2 + (\frac{\partial \hat{\varphi}_1}{\partial \hat{z}})^2\Big)](\tilde{x}, \hat{y}, \hat{\eta}_1) + \frac{\hat{\eta}_1}{\hat{Fr}^2} = 0 \tag{9}$$

The resulting problem (3), (5),(7),(9) can be physically interpreted as a deformable body (the variable $\tilde{x}$ represents the time) that generates two dimensionnal non linear waves. However, since the proposed solution only satisfies the two-dimensional Laplace equation, it is obviously not valid far from the ship. Hence, boundary conditions at infinity must be replaced by a matching condition with an outer solution.

3 Outer solution

The outer variables are based on the length scale L and velocity scale U:

$$\tilde{r} = \epsilon\, \hat{r}, \quad \tilde{y} = \epsilon\, \hat{y}, \quad \tilde{z} = \epsilon\, \hat{z}, \quad \tilde{\varphi} = \hat{\varphi}, \quad \tilde{\eta} = \hat{\eta} \tag{10}$$

and the following asymptotic expansions are assumed:

$$\tilde{\varphi}(\tilde{x}, \tilde{r}, \theta, \epsilon) = \tilde{\mu}_1(\epsilon)\, \tilde{\varphi}_1(\tilde{x}, \tilde{r}, \theta) + o(\tilde{\mu}_1), \qquad \tilde{\eta}(\tilde{x}, \tilde{y}, \epsilon) = \tilde{\nu}_1(\epsilon)\, \tilde{\eta}_1(\tilde{x}, \tilde{y}) + o(\tilde{\nu}_1) \tag{11}$$

The outer perturbation potential satisfies three-dimensional Laplace equation. The body boundary condition is replaced by a matching condition with inner solution. The free surface conditions are discussed taking into account the relative orders of magnitude of $\tilde{\mu}_1, \tilde{\nu}_1$ and Fr. It appears that only one case is acceptable for high Froude number. The dynamic free surface conditions reduces to:

$$\frac{\partial \tilde{\varphi}_1}{\partial \tilde{x}}(\tilde{x}, \tilde{y}, 0) = 0 \qquad (12)$$

This leads to $\tilde{\varphi}_1(\tilde{x}, \tilde{y}, 0) = 0$ everywhere on the linearized free surface except behind the ship, in the wake where $\tilde{\varphi}_1(\tilde{x}, 0, 0) = \tilde{\varphi}_1(1, 0, 0)$ (x=1 corresponds to the stern). The kinetic free surface condition yields:

$$[\frac{\partial \tilde{\eta}_1}{\partial \tilde{x}} - \frac{\partial \tilde{\varphi}_1}{\partial \tilde{z}}](\tilde{x}, \tilde{y}, 0) = 0 \qquad (13)$$

As Ogilvie [3], the solution of this problem is deduced from Ward's solution [5] proposed in aerodynamic for linearized high-speed flows past slender bodies:

$$\tilde{\varphi}_1(\tilde{x}, \tilde{r}, \theta) = \sum_{n=1}^{\infty} \frac{\sin(n\,\theta)}{\tilde{r}^n} \int_{-\infty}^{\infty} \frac{[\sqrt{(\tilde{x}-\xi)^2 + \tilde{r}^2} + \tilde{x} - \xi]^n}{\sqrt{(\tilde{x}-\xi)^2 + \tilde{r}^2}} f_n(\xi)\, d\xi \qquad (14)$$

This solution satisfies the three-dimensionnal Laplace equation and vanishes on the free-surface except on the x-axis where it is not yet defined. Functions f_n will be determine by matching.

4 Intermediate solution and matching

Since the inner problem can not be solved analytically, we must introduce an intermediate domain in order to match the inner and outer solutions. Kaplun's extension theorem [2] asserts that the range of validity of the inner or outer limits extends at least slightlty into the intermediate range. The intermediate variables are defined by:

$$\bar{r} = \frac{\tilde{r}}{\zeta(\epsilon)}, \quad \bar{y} = \frac{\tilde{y}}{\zeta(\epsilon)}, \quad \bar{z} = \frac{\tilde{z}}{\zeta(\epsilon)}, \quad \bar{\eta} = \tilde{\eta}, \quad \bar{\varphi} = \tilde{\varphi}, \quad with \quad \epsilon \ll \zeta(\epsilon) \ll 1. \qquad (15)$$

An asymptotic expansion of the solution is assumed and carrying out the intermediate limit in the free surface conditions and Laplace equation, we obtain the intermediate problem. The intermediate solution has to satisfy the two-dimensionnal Laplace equation and must vanish on the linearized free-surface. The solution is found by extending the harmonic potential to the whole domain and by using Laurent's expansion to develop the analytic function obtained. The solution for the intermediate perturbation potential is:

$$\bar{\varphi}_1(\bar{x}, \bar{r}, \theta) = \sum_{n=1}^{\infty} \sin(n\,\theta)\,[\frac{A_{1n}(\bar{x})}{\bar{r}^n} + B_{1n}(\bar{x})\,\bar{r}^n] \qquad (16)$$

Matching this solution, at first order, with the outer solution leads to:

$$\tilde{\varphi}_1(\tilde{x}, \tilde{r}, \theta) = \int_0^{\infty} -\frac{\mu(\xi)}{4\pi} \frac{\tilde{r}\,\sin(\theta)}{[(\tilde{x}-\xi)^2 + \tilde{r}^2]^{\frac{3}{2}}} d\xi \quad with \quad \mu(\tilde{x}) = -4\,\pi \int_0^{\tilde{x}} f_1(\xi)\, d\xi \qquad (17)$$

The outer solution is represented by a vertical three-dimensional dipoles distribution on the x-axis. In front of the ship, the perturbation potential vanishes. In the wake, the potential is constant and equal to its value on the stern. Since the jump in potential is equal to the intensity of the dipole, μ is also constant and equal to its value on the stern. This is equivalent imposing a Kutta condition. Indeed, any discontinuity in μ would result in a discrete vortex, of strength equal to the jump in the doublet distribution, along the trailing edge and would produce an infinite velocity at the edge. Consequently:

$$\tilde{\varphi}_1(\tilde{x},\tilde{r},\theta) = \int_0^1 -\frac{\mu(\xi)}{4\pi}\frac{\tilde{r}\sin(\theta)}{[(\tilde{x}-\xi)^2+\tilde{r}^2]^{\frac{3}{2}}}d\xi - \frac{\mu(1)}{4\pi}\frac{\sin(\theta)}{\tilde{r}}(1+\frac{\tilde{x}-1}{\sqrt{(\tilde{x}-1)^2+\tilde{r}^2}}) \tag{18}$$

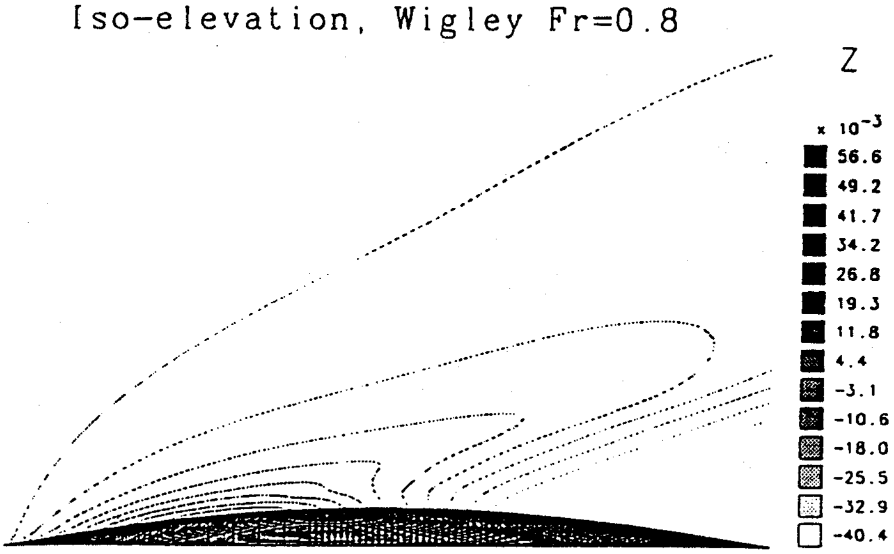
Matching the intermediate and inner solutions gives the order of magnitude of the outer solution $\tilde{\mu}_1(\epsilon)=\epsilon^3$, $\tilde{\nu}_1(\epsilon)=\epsilon^2$ and the behaviour of the inner solution at infinity:

$$\lim_{\hat{r}\to\infty}\hat{\varphi}_1(\tilde{x},\hat{r},\theta) = -\frac{\mu(\tilde{x})}{2\pi}\cdot\frac{\sin(\theta)}{\hat{r}} \tag{19}$$

The inner solution behaves like a two dimensionnal vertical dipole whose intensity $\mu(\tilde{x})$ is equal to the tridimensional dipole intensity of the outer solution. The non linear inner problem is well posed and can be solved numericaly for $\hat{\varphi_1}$, $\hat{\eta}_1$ and μ using the Sindbad code [1].

5 Numerical results

Results will be provided concerning the wave field and wave resistance of a serie 64 high-speed displacement form.



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DISCUSSION

Yue D.K.P: This concerns how the matching condition is applied to the inner solution. Since the inner problem is an initial-boundary-value problem starting from quiescent conditions, the solution vanishes with distance for finite time although the net contribution of the far-field boundary integral may not vanish, in general. If the matching condition is applied numerically at large distance, the homogenous Robin matching condition would have no effect. On the other hand, an analytic evaluation of the far-field contribution does not appear to benefit from the Robin condition. How does your total solution depend on the matching?

Fontaine E.: The Robin matching condition is applied, at a distance equal to the length of the ship. I have also imposed the condition $\phi = 0$ at the same distance and it appears that the free surface elevation differs slightly in the two simulations. These differences vanish when the condition is applied at a larger distance. It also appears that there are less instabilities in the jet with the use of this condition.

Tuck E.O: I simply want to know the attention of the authors to the fact that Ogilvie (1967) already has most of the matching details, including an outer flow which is a line of dipoles. I recall pointing out to Ogilvie at the time that this meant that one can ignore the outer flow, since all it does is provide the far-field condition $\phi \to 0$ (like a 2D dipole), for use in numerical solution of the inner problem. Of course, we did not have enough computer power in 1967 to solve this (nonlinear) inner problem, and it is good to see it being done now by the present (and previous paper) authors.

Fontaine E.: Indeed, our approach follows closely that of Ogilvie, but we use an asymptotic expansion rather than an asymptotic sequence. This means we must not assume the different terms remain ordered during the matching process. We also explicitly write a Kutta condition at the stern, so that the dynamic free surface condition is satisfied on the axis and the wake is determined.

Yeung R.W.: This type of slender-body theory has the puzzling feature that 1) the initial condition at the bow is not clear (i.e. aside from simply setting $\phi = \phi_t = 0$); 2) the role played by the transverse waves. Some of the answers to these issues may be found in the Hamburg ONR Symposium paper (Yeung & Kim, 1984), in which we showed the appropriate formulation for the inner problem needed to achieve a uniform behavior as $U \to 0$ or $\omega \to 0$.

Fontaine E.: Indeed, one can question the validity of some assumptions on which this theory is based on at the bow. From a practical point of view, the simulation can be started using a self-similar asymptotic solution. Concerning the transverse waves, measurements show that the role played by these waves decreases as the Froude number increases.