Application of the Flux Difference Splitting Method to Compute Nonlinear Shallow Water Flow on Deck

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Introduction

Nonlinear shallow water flow on deck due to ship motions and the resulted pressure on the deck are the subject of the present study. In this problem, a steep wave front or a bore has been observed in both experiment and numerical computation [1]. Mathematically, a discontinuity exists at the bore [2][3]. This brings difficulties to establish a stable numerical scheme and to capture the discontinuity. The classical schemes produced numerical results with fluctuated behavior near the discontinuity. To obtain a stable scheme, Steger and Warming [4] introduced the flux vector splitting method to compute the shock wave. The flux vector in the governing equations of gas dynamics was split based on the characteristic directions. This method takes the advantages of both of the finite difference method and the characteristic method. Since the nonhomogeneous governing equations for nonlinear shallow water flow on deck are different from those in gas dynamics, the flux difference operator, instead of the flux vector, is split in the present study. The Superbee flux limiter [2] has been employed in our algorithm. This makes the finite difference scheme of second-order with high resolution. The flux difference splitting method for the two-dimensional shallow water flow can also be extended to the three-dimensional case.

Nonlinear Shallow Water Flow on Deck due to Ship Motions

A ship-fixed coordinate system is used, see Fig. 1. Suppose that the ship oscillates with the sway acceleration a_2, heave acceleration a_3 and angular velocity in roll Ω_1 with respect to the coordinate system. The continuity equation and Euler's equations of motion can be written as follows:

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-gsin\phi-a_2+\Omega_1^2y+2\Omega_1w+(z-s)\dot{\Omega}_1 \tag{2}$$

$$\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-gcos\phi-a_3+(z-s)\Omega_1^2-y\dot{\Omega}_1-2\Omega_1v \tag{3}$$

where v and w are the velocity components in the y- and z- directions, respectively; p is the pressure; ϕ is the roll angle and s is the vertical coordinate of the roll axis. On the free surface, the kinematic condition is:

$$w=\frac{\partial\zeta}{\partial t}+v\frac{\partial\zeta}{\partial y}\quad \text{on } z=\zeta \tag{4}$$

and the dynamic free surface condition is:

$$p=0 \quad \text{on} \quad z=\zeta \tag{5}$$

On the surface of the deck well, the boundary conditions are:

$$v=0 \ \text{ on } \ y=\pm L, \quad \text{and} \quad w=0 \ \text{ on } \ z=0 \tag{6}$$

where L is the half-breadth of the deck. Using the shallow water assumption and v being a small quantity, the pressure can be obtained by integrating (3) with respect to z:

$$p(x,y,t) = \rho(g cos\phi + a_3 + x\Omega_1 + 2\Omega_1 v)(\zeta - z) - \frac{1}{2}\rho\dot{\Omega}_1^2[(\zeta - s)^2 - (z - s)^2] \quad (7)$$

The exciting forces on the ship caused by the water flow on deck can be obtained by integrating the pressure over the wetted deck area. Integrating the equations (1) and (2) from $z = 0$ to $z = \zeta$, the governing equations can be derived as follows:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(v\zeta) = 0 \quad (8)$$

$$\frac{\partial}{\partial t}(u\zeta) + \frac{\partial}{\partial x}(u^2\zeta + \frac{1}{2}\zeta^2) = Q_1\zeta^2\frac{\partial \zeta}{\partial x} + Q_2 u\zeta\frac{\partial \zeta}{\partial y} + Q_3\zeta\frac{\partial \zeta}{\partial y} + Q_4\zeta \quad (9)$$

where $Q_1 = \dot{\Omega}_1$, $Q_2 = -2\Omega_1$, $Q_3 = (1 - cos\phi)g - a_3 - x\dot{\Omega}_1 - s\dot{\Omega}_1$, and $Q_4 = -g sin\phi - a_2 + x\Omega_1^2 - s\dot{\Omega}_1$. The governing equations (8) and (9) can also be expressed in the vector form:

$$\frac{\partial \vec{W}}{\partial t} + \frac{\partial \vec{G}}{\partial y} = [D]\frac{\partial \vec{H}}{\partial y} + \vec{C} \quad (10)$$

in which, $\vec{W} = \begin{pmatrix} u \\ u\zeta \end{pmatrix}$, $\vec{G} = \begin{pmatrix} u\zeta \\ u^2\zeta + \frac{1}{2}\zeta^2 \end{pmatrix}$, $\vec{H} = \begin{pmatrix} 0 \\ \frac{1}{2}\zeta^2 \end{pmatrix}$, $[D] = \begin{pmatrix} 0 & 0 \\ 0 & Q_1\zeta + Q_2 v + Q_3 \end{pmatrix}$, and $\vec{C} = \begin{pmatrix} 0 \\ Q_4\zeta \end{pmatrix}$. The derivative of the flux vector $\vec{G}$ and $\vec{H}$ can be expressed in terms of vector $\vec{W}$, let $\vec{F} = \vec{G} + [D]\vec{H}$,

$$\frac{\partial \vec{F}}{\partial y} = \frac{\partial \vec{G}}{\partial y} - [D]\frac{\partial \vec{H}}{\partial y} = [J]\frac{\partial \vec{W}}{\partial y} \quad (11)$$

where the Jacobian matrix $[J] = \frac{\partial \vec{F}}{\partial \vec{W}}$. The eigenvalues of the Jacobian matrix are:

$$\lambda_1 = v + \psi, \quad \text{and} \quad \lambda_2 = v - \psi \quad (12)$$

with the eigenvectors:

$$\vec{e}_1 = (1, v + \psi)^T, \quad \text{and} \quad \vec{e}_2 = (1, v - \psi)^T \quad (13)$$

where $\psi = \sqrt{Q\zeta}$ and $Q = Q_1 v + Q_2\zeta + Q_3 + 1$. The charicteristic directions are determined by the above eigenvalues.

Flux Difference Splitting Scheme

From the equations in the previous section, $[\tilde{J}]$, the approximate Jacobian matrix, is so constructed that

$$\Delta\vec{F} = [\tilde{J}]\Delta\vec{W} \quad (14)$$

The difference of $\vec{W}$ and $\vec{F}$ are approximated in the present study as follows:

$$\Delta\vec{W} = \sum_{i=1}^{2}\alpha_i\vec{e}_i, \quad \text{and} \quad \Delta\vec{F} = \sum_{i=1}^{2}\lambda_i\alpha_i\vec{e}_i = [\tilde{J}]\Delta\vec{W} \quad (15)$$

and the source term $\vec{C}$ is projected into the eigenvector space:

$$\vec{C} = -\frac{1}{\Delta x}\sum_{i=1}^{2}\lambda_i\gamma_i\alpha_i \tag{16}$$

If the eigenvectors are split into $\lambda_i = \lambda_i^+ + \lambda_i^-$, for $i = 1, 2$, with $\lambda_i^\pm = \frac{1}{2}(\lambda_i \pm |\lambda_i|)$, the flux difference can be split corresponding to the eigenvectors, and a second-order scheme is obtained:

$$\vec{W}_j^{n+1} = \vec{W}_j^n - \beta\Delta\vec{F}_{j-\frac{1}{2}}^+ - \beta\Delta\vec{F}_{j+\frac{1}{2}}^- \tag{17}$$

where

$$\begin{aligned}\Delta\vec{F}_{j-\frac{1}{2}}^+ &= \frac{1}{2}(\vec{F}_j - \vec{F}_{j-1}) + \frac{1}{2}\sum_{i=1}^{2}\lambda_{i,j-\frac{1}{2}}\gamma_{i,j-\frac{1}{2}}\vec{e}_{i,j-\frac{1}{2}} + \frac{1}{2}\sum_{i=1}^{2}(\alpha_{i,j-\frac{1}{2}} + \lambda_{i,j-\frac{1}{2}}\gamma_{i,j-\frac{1}{2}})|\lambda_{i,j-\frac{1}{2}}|\vec{e}_{i,j-\frac{1}{2}} \\ &+\frac{1}{2}\sum_{i=1}^{2}\phi_{j-\frac{1}{2}}(\alpha_{i,j-\frac{1}{2}} + \lambda_{i,j-\frac{1}{2}}\gamma_{i,j-\frac{1}{2}})|\lambda_{i,j-\frac{1}{2}}|(1-\beta|\lambda_{i,j-\frac{1}{2}}|)\vec{e}_{i,j-\frac{1}{2}}\end{aligned} \tag{18}$$

$$\begin{aligned}\Delta\vec{F}_{j+\frac{1}{2}}^- &= \frac{1}{2}(\vec{F}_{j+1} - \vec{F}_j) + \frac{1}{2}\sum_{i=1}^{2}\lambda_{i,j+\frac{1}{2}}\gamma_{i,j+\frac{1}{2}}\vec{e}_{i,j+\frac{1}{2}} - \frac{1}{2}\sum_{i=1}^{2}(\alpha_{i,j+\frac{1}{2}} + \lambda_{i,j+\frac{1}{2}}\gamma_{i,j+\frac{1}{2}})|\lambda_{i,j+\frac{1}{2}}|\vec{e}_{i,j+\frac{1}{2}} \\ &-\frac{1}{2}\sum_{i=1}^{2}\phi_{j+\frac{1}{2}}(\alpha_{i,j+\frac{1}{2}} + \lambda_{i,j+\frac{1}{2}}\gamma_{i,j+\frac{1}{2}})|\lambda_{i,j+\frac{1}{2}}|(1-\beta|\lambda_{i,j+\frac{1}{2}}|)\vec{e}_{i,j+\frac{1}{2}}\end{aligned} \tag{19}$$

The above two equations show that $\Delta\vec{F}_{j-\frac{1}{2}}^+$ is approximated by a backward finite difference, and $\Delta\vec{F}_{j+\frac{1}{2}}^-$ by a forward finite difference. Both follow the characteristic directions of wave propgation. $\phi_{j\pm\frac{1}{2}}$ is the so-called Superbee limiter and it is a correction term for the numerical flux. Without this limiter, the finite difference scheme would become first-order.

Results and Discussions

The shallow water flow in a deck well of 0.91^m long is calculated. The mean water depth is 0.05^m. The deck is forced to oscillate in a roll motion with an amplitude of $7.5^{\,deg}$ and a frequency $4.712^{rad/sec}$. The results are shown in Fig.2 to Fig.4. A bore can be clearly observed in Fig. 2 and Fig.4. The numerical results are also compared with the experimental data from Ref. [1]. The effect of the Superbee limiter can be seen in Fig.5, where the dotted curve is the results without the limiter. The jump of the water surface near the bore is smeared out from the first-order scheme. By including the Superbee limiter, numerical results with high resolution can be obtained. For instance, from the computation of a typical bore propagation problem, the discontinuity can be captured within two discretized segments. Fig. 6 shows the result of the water flow in a deck well of 0.62^m long. The water depth is 0.05^m and the ship has the sway motion only. The sway frequency is $3.6^{rad/sec}$, very close to the resonant frequency, and the amplitude is 0.015^m. A bore is generated after one forced oscillatory period and moves back and forth in the deck well. As the frequency increases, two bores can be observed and they propagate in the opposite directions. A high wave elevation occurs when they meet each other. At the high frequency, very short waves are observed. In all above computations, the Courant number is less than 1.0.

The flux difference splitting method has currently been extended to the 3-dimensional case. The numerical scheme with the consideration of ship motions in six degrees of freedom has been developed. For illustration, the computed water wave motion on deck due to surge is shown in Fig.7, and due to surge and sway is shown in Fig.8. More results will be presented in the Workshop.

References

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3. Stoker, J. J.,"Water Waves", Interscience Publishers, Inc., New York, 1957.
4. Steger, J.L. & Warming, R. F., J. Comp. Physics, Vol. 40, 1981.

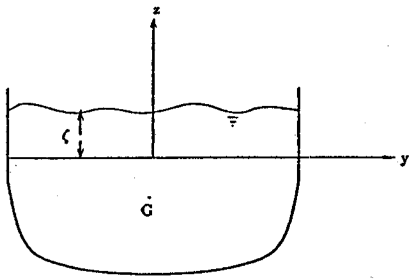

Fig.1 The coordinate system

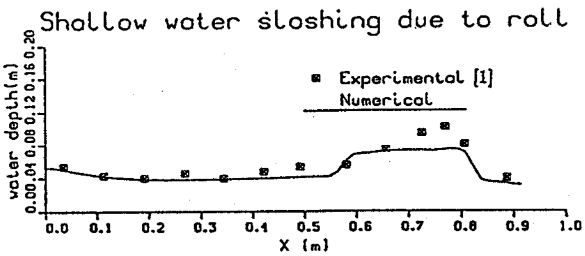

Fig.2 Wave motion at $t = 1.0$ sec.

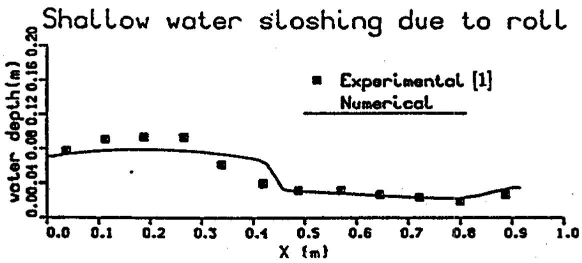

Fig.3 Wave motion at $t = 2.0$ sec.

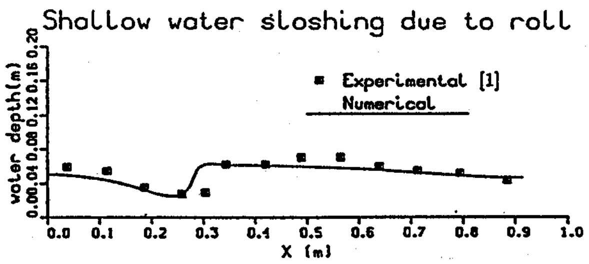

Fig.4 Wave Motion at $t = 3.0$ sec.

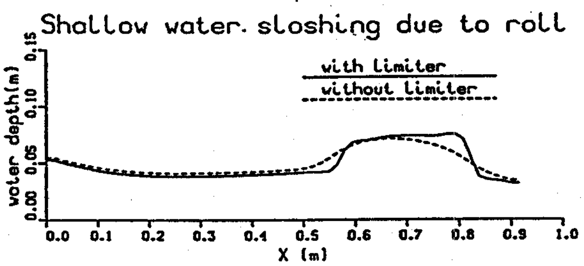

Fig.5 Effect of the Superbee limiter ($t = 3.0$ sec.)

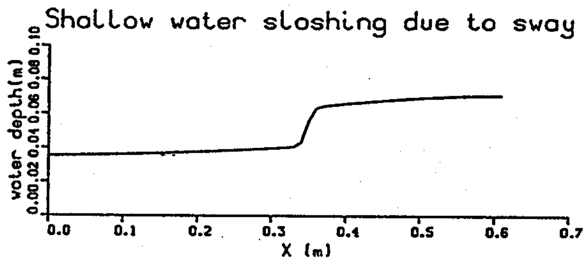

Fig.6 Wave motion at $t = 3.0$ sec.

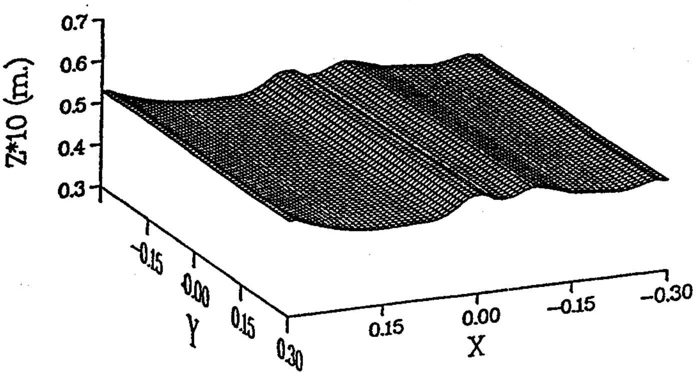

Fig.7 The 3-D wave motion due to surge.

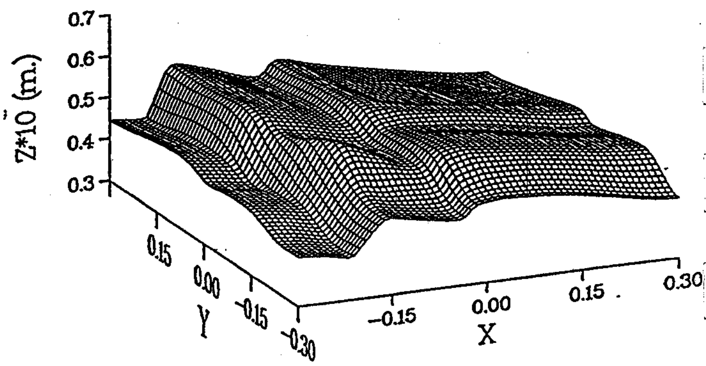

Fig.8 The 3-D wave motion due to surge and sway

DISCUSSION

Armenio V.: Have you analyzed the coupling between roll motion of the ship and the oscillation of the water on deck in several conditions of excitation?

Huang Z.J. & Hsiung C.C.: In the present work, the water flow on deck was computed only for the known ship motions.

Armenio V.: In the past, the Glimm's method had been used for the solution to this problem. You know that this method suffers for a strong mass variation in the time-domain simulation. In the past at my department, we have used a Lax-Wendroff hybridized method, and at the moment the "Method of Space-Time Conservation Element and Solution Element" by Chan and To, and the previous problem has been drastically reduced. Did you experience this kind of problem?

Huang Z.J. & Hsiung C.C: The mass variation is very small in our computation when the Superbee Limiter is adopted. For example, after 20 cycles of deck oscillation, the mass variation is less than 0.5%.