THIRD ORDER TRIPLE FREQUENCY WAVE FORCES ON FIXED VERTICAL CYLINDERS

Š.MALENICA & B. MOLIN
Institut Français du Pétrole, BP 311
92506 Rueil-Malmaison, France

Introduction

In order to better understand the so-called "ringing" phenomenon, observed on some offshore structures in the North Sea, we try to calculate the third harmonic of the forces acting on a vertical cylinder in regular waves. The method that we use is based on the ring source method [1][2][3][4][5] with the Green function expressed as a series of eigenfunctions. Since this method is the basis of the present work we will explain it briefly.

Consider the standard hydrodynamic problem of finding a potential wich satisfies the Laplace equation in the fluid, zero normal derivative on the fixed boundary, a radiation condition, and the inhomogeneous free surface condition :

$$-\alpha\varphi + \frac{\partial\varphi}{\partial z} = Q(r,\theta)$$

The Green function of the problem which satisfies the following set of equations :

$$\begin{array}{ll}
\Delta G(\mathrm{x},\xi) = -4\pi\delta(\mathrm{x}) & 0 \leq \zeta \leq H \\
-\alpha G + \frac{\partial G}{\partial \zeta} = 0 & \zeta = 0 \\
\frac{\partial G}{\partial \zeta} = 0 & \zeta = -H \\
\lim\left[\sqrt{k_0\rho}\left(\frac{\partial G}{\partial \rho} - ik_0 G\right)\right] = 0 & \rho \to \infty
\end{array}$$

can be written in the form of the Fourier series :

$$G(\mathrm{x},\xi) = \sum_{m=0}^{\infty} \epsilon_m g_m(r,z;\rho,\zeta)\cos m(\theta - \vartheta)$$

with :

$$g_m(r,z;\rho,\zeta) = -2\pi i C_0 \begin{pmatrix} H_m(k_0 r)J_m(k_0\rho) \\ J_m(k_0 r)H_m(k_0\rho) \end{pmatrix} f_0(z)f_0(\zeta) + 4\sum_{n=1}^{\infty} C_n \begin{pmatrix} K_m(k_n r)I_m(k_n\rho) \\ I_m(k_n r)K_m(k_n\rho) \end{pmatrix} f_n(z)f_n(\zeta) \qquad \begin{pmatrix} r > \rho \\ r < \rho \end{pmatrix}$$

where k_0 and k_n follow from $\alpha = k_0 \tanh k_0 H = -k_n \tan k_n H$ and :

$$f_0(z) = \frac{\cosh k_0(z+H)}{\cosh k_0 H} \quad , \quad f_n(z) = \frac{\cos k_n(z+H)}{\cos k_n H} \quad , \quad C_0 = \left[2\int_{-H}^{0} f_0^2(z)dz\right]^{-1} \quad , \quad C_n = \left[2\int_{-H}^{0} f_n^2(z)dz\right]^{-1}$$

Potential and forcing term on the free surface are also expressed in the form of Fourier series and the standard integral equation is written :

$$\begin{pmatrix} 4\pi\varphi(\mathrm{x}) \\ 0 \end{pmatrix} - \int_{S_{B0}} \varphi(\xi)\frac{\partial G(\mathrm{x},\xi)}{\partial \rho}dS = \int_{S_F} G(\mathrm{x},\xi)Q(\rho,\vartheta)dS + \int_{S_\infty}\left[G(\mathrm{x},\xi)\frac{\partial\varphi(\xi)}{\partial\rho} - \varphi(\xi)\frac{\partial G(\mathrm{x},\xi)}{\partial\rho}\right]dS \qquad \begin{pmatrix} r > a \\ r < a \end{pmatrix}$$

This equation is considerably simplified if the last integral disappears. That depends on the behavior of the potential φ at infinity and we will see later that this is true for the cases considered here.
If we suppose the Fourier coefficients of the potential φ, on the cylinder, in the form:

$$\varphi_m(a,z) = f_0(z)A_{m0} + \sum_{n=1}^{\infty} f_n(z)A_{mn}$$

and if we write the integral equation for $r = a - \delta, (\delta > 0)$ we obtain the following expressions for the A_{mn} coefficients :

$$A_{m0} = -\frac{2C_0 \int_a^\infty H_m(k_0\rho)Q_m(\rho)\rho d\rho}{k_0 a H'_m(k_0 a)}, \qquad A_{mn} = -\frac{2C_n \int_a^\infty K_m(k_n\rho)Q_m(\rho)\rho d\rho}{k_n a K'_m(k_n a)}$$

Knowing the potential on the cylinder we can find the potential everywhere in the fluid :

$$\begin{aligned}
\varphi_m(r,z) \;=\; & \pi i C_0 f_0(z) H_m(k_0 r) \int_a^r [J_m(k_0\rho) - Z_{m0} H_m(k_0\rho)] Q_m(\rho)\rho d\rho \\
& + 2\sum_{n=1}^{\infty} C_n f_n(z) K_m(k_n r) \int_a^r [I_m(k_n\rho) - Z_{mn} K_m(k_n\rho)] Q_m(\rho)\rho d\rho \\
& + \pi i C_0 f_0(z) [J_m(k_0 r) - Z_{m0} H_m(k_0 r)] \int_r^\infty H_m(k_0\rho) Q_m(\rho)\rho d\rho \\
& + 2\sum_{n=1}^{\infty} C_n f_n(z) [I_m(k_n r) - Z_{mn} K_m(k_n r)] \int_r^\infty K_m(k_0\rho) Q_m(\rho)\rho d\rho
\end{aligned}$$

With $Z_{m0} = J'_m(k_0 a)/H'_m(k_0 a)$ and $Z_{mn} = I'_m(k_n a)/K'_m(k_n a)$.
The expression obtained in this way is exactly the same as in [1] where the authors use a special kind of Green function which satisfies the homogeneous condition on the cylinder.

Potentials

By employing the usual manner of linearisation we express the total potential in a perturbation serie $\phi = \varepsilon\varphi^{(1)} + \varepsilon^2\varphi^{(2)} + \varepsilon^3\varphi^{(3)} + \ldots$ and we transform the difficult nonlinear boundary value problem to the sum of the different boundary value problems corresponding to the different orders of approximation.
The coefficient α for the first order problem is ν, for the second order problem is 4ν and for third order problem 9ν, where $\nu = \omega^2/g$ is the wavenumber in the infinite water depth. Corresponding forcing terms on the free surface are:

$$\begin{aligned}
Q^{(1)}(r,\theta) &= 0 \\
Q^{(2)}(r,\theta) &= \frac{i\omega}{g}\nabla\varphi^{(1)}\nabla\varphi^{(1)} - \frac{i\omega}{2g}\varphi^{(1)}(\varphi^{(1)}_{zz} - \nu\varphi^{(1)}_z) \\
Q^{(3)}(r,\theta) &= \frac{3i\omega}{g}\nabla\varphi^{(2)}\nabla\varphi^{(1)} - \frac{i\omega}{2g}[\varphi^{(1)}(\varphi^{(2)}_{zz} - 4\nu\varphi^{(2)}_z) + 2\varphi^{(2)}(\varphi^{(1)}_{zz} - \nu\varphi^{(1)}_z)] \\
& \quad - \frac{1}{8g}\nabla\varphi^{(1)}\nabla(\nabla\varphi^{(1)}\nabla\varphi^{(1)}) - \frac{\nu}{g}\varphi^{(1)}\nabla\varphi^{(1)}\nabla\varphi^{(1)}_z \\
& \quad + \frac{1}{4g}(\nu\varphi^{(1)}\varphi^{(1)}_z + \frac{1}{2}\nabla\varphi^{(1)}\nabla\varphi^{(1)})(\varphi^{(1)}_{zz} - \nu\varphi^{(1)}_z)
\end{aligned}$$

In the potential of each order we must distinguish three parts $\varphi = \varphi_I + \varphi_{DI} + \varphi_D$. The incident potential φ_I is easy to find. The first part of diffracted potential φ_{DI} is also relatively easy to calculate because it is in fact the standard diffraction potential with the normal velocity on the body which opposes the velocity induced by the incident potential, and homogeneous condition on the free surface. The second part of the diffracted potential φ_D represents the main difficulty of calculation because it satisfies the inhomogeneous condition on the free surface (in which the part originating only from the incident potentials has been

eliminated). On the cylinder it satisfies the homogeneous condition and the method presented in the begining of this paper can be applied to calculate this part of the potential.
As we have seen first of all we must make sure that the integral on the control surface at infinity disappears. To prove this we must study the behavior of the potential at infinity. First order problem is trivial and well known. We also know [6] the expression at infinity for the second order diffraction potential in which we can distinguish locked and free wave components:

$$\varphi_D^{(2)} = \frac{g_1(\theta)}{\sqrt{R}}\frac{\cosh K_1(z+H)}{\cosh K_1 H}e^{ik_0R(1+\cos\theta)} + \frac{g_2(\theta)}{\sqrt{R}}\frac{\cosh \kappa_0(z+H)}{\cosh \kappa_0 H}e^{i\kappa_0 R} + O(\frac{1}{R})$$

The expression for the third order diffraction potential at infinity can be found in the same way as:

$$\begin{aligned}\varphi_D^{(3)} &= \frac{h_1(\theta)}{\sqrt{R}}\frac{\cosh K_2(z+H)}{\cosh K_2 H}e^{ik_0R(1+2\cos\theta)} + \frac{h_2(\theta)}{\sqrt{R}}\frac{\cosh K_3(z+H)}{\cosh K_3 H}e^{iR(\kappa_0+k_0\cos\theta)} \\ &+\frac{h_3(\theta)}{\sqrt{R}}\frac{\cosh \mu_0(z+H)}{\cosh \mu_0 H}e^{i\mu_0 R} + O(\frac{1}{R})\end{aligned}$$

With $\nu = k_0 \tanh k_0 H = \frac{1}{4}\kappa_0 \tanh \kappa_0 H = \frac{1}{9}\mu_0 \tanh \mu_0 H$ and:

$$K_1 = k_0\sqrt{2(1+\cos\theta)} \quad , \quad K_2 = k_0\sqrt{5+4\cos\theta} \quad , \quad K_3 = \sqrt{k_0^2+\kappa_0^2+2k_0\kappa_0\cos\theta}$$

As we can see, for the third order potential we have a free wave component and two locked waves. Anyway, in all cases the behavior is in $O(R^{-1/2})$ and, by employing the stationary phase method, we can show that the integral disappears at infinity.
For the calculation of the forcing term on the free surface in the third order problem we need to know the second order potential on the free surface and also its derivatives. In its calculation special attention must be given to the treatement of the singularity which occurs on the free surface. This problem was treated by employing a method similar to that in [1] in combination with an iterative procedure for the given tolerance. Also, by employing some integral equality, we can avoid the calculation of either the double derivative with respect to z or the derivative with respect to r of the second order potential, in the free surface condition for $\varphi_D^{(3)}$. In our calculation we prefered to eliminate the double z derivative. Very important problem is also the oscillatory infinite integral which appears in the expression for the coefficients A_{m0}. In the second order problem this integral can be calculated semi-analytically [1][2] but in the third order problem it is not the case, because we have no analytical expression for $\varphi_{Dm}^{(2)}$, and we must calculate this integral numerically until convergence.

Forces

By introducing the perturbation series for the potential in the expression for the forces we obtain the following expression for the corresponding orders :

$$\begin{aligned}F^{(1)} &= i\omega\varrho\int_{S_{B0}}\varphi^{(1)}\mathbf{n}dS \\ F^{(2)} &= \int_{S_{B0}}(2i\omega\varphi^{(2)} - \frac{1}{4}\varrho\nabla\varphi^{(1)}\nabla\varphi^{(1)})\mathbf{n}dS + \frac{1}{4}\varrho g\int_{C_{B0}}\eta^{(1)}\eta^{(1)}\mathbf{n}dC \\ F^{(3)} &= \int_{S_{B0}}(3i\omega\varrho\varphi^{(3)} - \frac{1}{2}\varrho\nabla\varphi^{(1)}\nabla\varphi^{(2)})\mathbf{n}dS + \frac{1}{2}\varrho g\int_{C_{B0}}\eta^{(1)}(\eta^{(2)} - \frac{\nu}{4}\eta^{(1)}\eta^{(1)})\mathbf{n}dC\end{aligned}$$

where $\eta^{(1)}$ and $\eta^{(2)}$ are the free surface elevations on the body:

$$\begin{aligned}\eta^{(1)} &= \frac{i\omega}{g}\varphi^{(1)} \\ \eta^{(2)} &= \frac{2i\omega}{g}\varphi^{(2)} - \frac{1}{4g}\nabla\varphi^{(1)}\nabla\varphi^{(1)} + \frac{\nu}{2}\eta^{(1)}\eta^{(1)}\end{aligned}$$

The normal $\mathbf{n}$ is pointing out of the fluid and, in our case, since the problem is symmetrical about x axis, we have $\mathbf{n} = n_x = -\cos\theta$. So we need to know only the first harmonic in the Fourier series of the function under the integral in the expression for the forces. However to calculate it, in the third order case, we must calculate the complete second order potential.

Results

For the second order quantities we obtain the same results as in [1][2]. Some preliminary results for the third order quantities are shown on figures 1 and 2. Figure 1 represents the $m = 1$ component of the forcing term on the free surface, in wich the double z derivative of $\varphi_D^{(2)}$ has been eliminated, and figure 2 represents the highly oscillatory integral which appears in expression for $A_{m0}^{(3)}$ and which is in fact the integral of the forcing term (fig.1) multiplied by $H_m(\mu_0\rho)\rho$.

References

1. Chau F.P., R.Eatock Taylor : "Second-order wave diffraction by a vertical cylinder",J.F.M.,vol.240, pp. 571-599, 1992.
2. Kim M.H., D.K.P.Yue : "The complete second-order diffraction solution for an axisymetric body, Part 1.Monochromatic incident waves",J.F.M.,vol.200, pp. 235-264, 1989.
3. Fenton J.D. :"Wave forces on vertical bodies of revolution",J.F.M., Vol.85, pp. 241-255, 1978.
4. Malenica Š., P.J.Clark :"Amortissement de dérive pour un groupe de cylindres en profondeur finie", Actes des 4èmes Journées de l'hydrodynamique, Nantes 1993.
5. Matsui T., L.S.Yeob et S.Kimitoshi : "Hydrodynamic forces on a vertical cylinder in current and waves", J.of the Soc. of Naval Arch. of Japan, Vol. 170, pp.277-287, 1991.
6. Molin B. : "Second order diffraction loads upon three dimensional bodies", Appl. Ocean Res. 1, pp. 197-202, 1979.

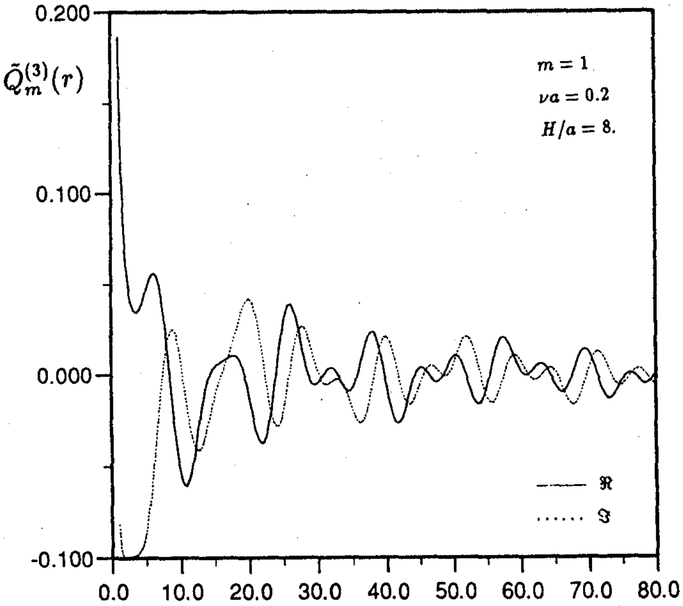

FIGURE 1.Real and imaginary part of the forcing term on the free surface for third order problem.

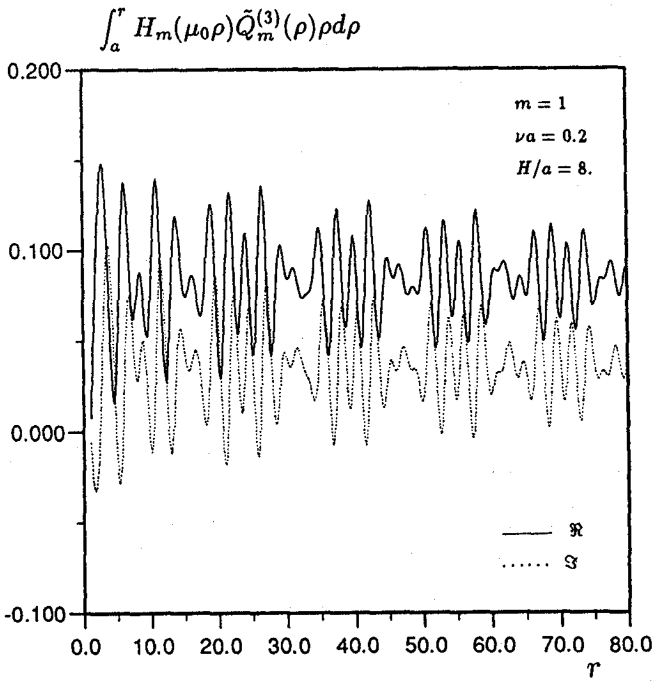

FIGURE 2.Real and imaginary part of the infinite oscillatory integral.