HYDRODYNAMIC FORCES ON A SUBMERGED CYLINDER ADVANCING IN WAVES OF TWO-LAYER FLUIDS

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The linear potential problem on movement of a submerged horizontal cylinder in regular waves at constant forward speed has been studied only in a case of homogeneous fluid [1-4]. To generalize this problem, let us consider the case of two-fluid system, both of the fluids being inviscid, incompressible, with constant, but different densities. The body moves below the interface, keeping constant time-averaged submergence and oscillates harmonically due to the action of following or head interfacial waves with the crests parallel to a cylinder axis.

For the sake of simplicity, let us suppose that the upper fluid is bounded by a rigid lid and the lower one is of infinite depth. The motion of a body under a free surface of the homogeneous fluid is a particular case of this problem.

Let the fixed coordinate system be taken with $\bar{x}$ -axis directed along an equilibrium position of the interface, orthogonally with respect to a cylinder axis, and y -axis pointed vertically upwards. In the undisturbed state, the upper layer, with the thickness H and density ρ_1, occupies the domain $-\infty < \bar{x} < \infty,\ 0 < y < H$, the lower one, with density $\rho_2 = \rho_1(1+\epsilon)$ $(\epsilon > 0)$, occupies the domain $-\infty < \bar{x} < \infty,\ y < 0$.

In the fixed frame of reference the incident potential may be written as

$$\psi_0^{(s)} = \frac{i\omega_0}{k_0}\phi_0^{(s)} \exp[i(\omega_0 t \mp k_0\bar{x})],$$

$$\phi_0^{(1)} = -\frac{\cosh k_0(y-H)}{\sinh k_0 H}, \quad \phi_0^{(2)} = e^{k_0 y},$$

where the incident-wave frequency ω_0 depends on wavenumber k_0 according to the dispersion relation,

$$\omega_0 = \Omega(k_0), \quad \Omega(k) = \sqrt{\epsilon g k F(k)}, \quad F(k) = 1 + \epsilon + \coth kH,$$

signs "+" and "-" correspond to waves travelling from right and from left, respectively, superscript s is equal to 1 for the upper layer and 2 for the lower one, g is the gravitational acceleration.

In the moving reference frame $x = \bar{x} - Ut$ the total potential can be written as

$$\Phi^{(s)}(x,y,t) = -Ux + U\bar{\Phi}^{(s)}(x,y) + \mathrm{Re} \sum_{j=o}^{4} \eta_j \Phi_j^{(s)}(x,y) e^{i\omega t},$$

where $\bar{\Phi}^{(s)}$ is the steady potential due to unit forward speed; the components $\Phi_j^{(s)}$ $(j = 1,2,3)$ are the radiation potentials due to motions of the cylinder with unit amplitude in each of three degrees of freedom; η_j $(j = 1,2,3)$ are corresponding motion amplitudes; $\Phi_0^{(s)} = \phi_0^{(s)} \exp(\mp ik_0 x)$ and $\Phi_4^{(s)}$ are the potentials of the incident and diffracted waves, respectively; and $\eta_0 = \eta_4$ is the incoming wave amplitude. In the moving reference frame, the incident waves arrive with the encounter frequency $\omega = \omega_0 \mp k_0 U$.

Based on the assumptions of linear potential flow theory, we can write the following governing equations for the steady potential

$$\Delta\bar{\Phi}^{(1)} = 0 \quad (0 < y < H), \quad \Delta\bar{\Phi}^{(2)} = 0 \quad (y < 0) \tag{1}$$

with boundary conditions

$$\partial\bar{\Phi}^{(1)}/\partial y = 0 \quad (y = H),$$

$$(1+\epsilon)\frac{\partial^2\bar{\Phi}^{(2)}}{\partial x^2} - \frac{\partial^2\bar{\Phi}^{(1)}}{\partial x^2} + \frac{\epsilon g}{U^2}\frac{\partial\bar{\Phi}^{(1)}}{\partial y} = 0, \quad \frac{\partial\bar{\Phi}^{(1)}}{\partial y} = \frac{\partial\bar{\Phi}^{(2)}}{\partial y} \quad (y = 0),$$

$$\frac{\partial\bar{\Phi}^{(2)}}{\partial y} \to 0 \ (y \to -\infty), \quad \frac{\partial\bar{\Phi}^{(s)}}{\partial x} \to 0 \ (x \to \infty), \quad \left|\frac{\partial\bar{\Phi}^{(s)}}{\partial x}\right| < \infty \ (x \to -\infty).$$

The cylinder is assumed to be located entirely in the lower layer and the following boundary condition for $\bar{\Phi}^{(2)}$ should be satisfied at mean position of body surface L

$$\partial\bar{\Phi}^{(2)}/\partial n = n_x \quad (x, y \in L),$$

where $\vec{n}$ is the inward normal of the cylinder surface and n_x is the component of $\vec{n}$ in the x -direction.

The radiation and diffraction components of potential satisfy the following equations, similar to (1):

$$\Delta\Phi_j^{(1)} = 0 \quad (0 < y < H), \quad \Delta\Phi_j^{(2)} = 0 \quad (y < 0)$$

with boundary conditions

$$\partial\Phi_j^{(1)}/\partial y = 0 \quad (y = H), \tag{2}$$

$$(1+\epsilon)D\Phi_j^{(2)} - D\Phi_j^{(1)} + \epsilon g \partial\Phi_j^{(1)}/\partial y = 0, \quad \partial\Phi_j^{(1)}/\partial y = \partial\Phi_j^{(2)}/\partial y \quad (y = 0), \tag{3}$$

$$\partial\Phi_j^{(2)}/\partial y \to 0 \quad (y \to -\infty), \tag{4}$$

$$\partial\Phi_j^{(2)}/\partial n = i\omega n_j - U m_j \ (j = 1, 2, 3), \quad \partial\Phi_4^{(2)}/\partial n = -\partial\Phi_0^{(2)}/\partial n \quad (x, y \in L),$$

where

$$D \equiv (U\partial/\partial x - i\omega)^2, \quad (n_1, n_2) = (n_x, n_y), \quad n_3 = (y - y_0)n_x - (x - x_0)n_y,$$

$$(m_1, m_2, m_3) = \left\{\frac{\partial^2\bar{\Phi}^{(2)}}{\partial n \partial x}, \frac{\partial^2\bar{\Phi}^{(2)}}{\partial n \partial y}, \frac{\partial}{\partial n}\left[(y - y_0)\left(\frac{\partial\bar{\Phi}^{(2)}}{\partial x} - 1\right) - (x - x_0)\frac{\partial\bar{\Phi}^{(2)}}{\partial y}\right]\right\}$$

x_0 and y_0 are the coordinates of a point with respect to which the body oscillates rotationally.

The radiation condition for $\Phi_j^{(s)}$ $(j = 1, ..., 4)$ states that a wave travelling in the direction of the forward speed and with its group velocity larger than the forward speed is far in front of the body, and otherwise the waves propagate behind.

After the steady and oscillatory potentials have been obtained, the pressure in the fluid can be determined from the Bernoulli equation. The hydrodynamic forces and moments can be obtained by integrating the pressure over the body surface. The steady, radiating and exciting loads are determined as for homogeneous fluid with a free surface

The coupled finite-element method [3] seems to be the most efficient numerical method for determination of hydrodynamic loads on a horizontal cylinder advancing in regular surface waves. This numerical method combines a finite-element approximation of the potential in a region surrounding the cylinder with a boundary-integral-equation representation in the outer region. This method can be applied in a case of stratified fluid, if density variation takes place only on the horizons above or below a submerged body.

To use the coupled finite-element method it is necessary to determine the Green function $G^{(s)}(x,y,\xi,\eta)$, satisfying the equations

$$\Delta G^{(1)} = 0 \quad (0 < y < H), \quad \Delta G^{(2)} = 2\pi\delta(x-\xi, y-\eta) \quad (y < 0)$$

and boundary conditions similar to (2)-(4). The solution of the problem for the Green function in the lower layer $G^{(2)}$ takes the form:

$$G^{(2)} = \ln(r\, r_1) + 2(1+\epsilon)\ \mathrm{pv} \int_0^\infty \frac{F(k)}{kP(k)} e^{k(y+\eta)} \times$$

$$\times \left\{ \left[(U^2k^2-\omega^2)^2 - (U^2k^2+\omega^2)\Omega^2(k) \right] \cos k(x-\xi) + 2i\omega k U \Omega^2(k) \sin k(x-\xi) \right\} dk +$$

$$+\pi\{\alpha_1 \exp[k_1(y+\eta - i(x-\xi))] - \alpha_2 \exp[k_2(y+\eta-i(x-\xi))] -$$

$$-\alpha_3 \exp[k_3(y+\eta+i(x-\xi))] + \alpha_4 \exp[k_4(y+\eta+i(x-\xi))]\},$$

where pv indicates the principal-value integration,

$$r^2 = (x-\xi)^2 + (y-\eta)^2, \quad r_1^2 = (x-\xi)^2 + (y+\eta)^2,$$

$$P = \prod_{s=1}^{4} P_s, \quad P_{1,2}(k) = Uk + \omega \mp \Omega(k),$$

$$P_{3,4}(k) = Uk - \omega \mp \Omega(k),$$

$$\alpha_s = \frac{i(1+\epsilon)\Omega(k_s)F(k_s)}{2k_s[U - \gamma c_g(k_s)]} \quad (\gamma = 1 \text{ at } s = 1,2,3 \text{ and } \gamma = -1 \text{ at } s = 4),$$

$c_g(k_s) = d\Omega/dk|_{k=k_s}$ is the group velocity of the wave k_s. The equation $P_1(k) = 0$ has two simple real solutions, k_1 and k_2, with $k_1 > k_2$, if only

$$U < U_c, \quad \omega < \omega_c, \tag{5}$$

where $U_c = \sqrt{\epsilon g H}$ is the critical velocity for a steady problem in the two-layer and $\omega_c = \Omega(k_c) - Uk_c$ is defined post-solving the equation $c_g(k_c) = U$. Solutions k_1 and k_2 coincide, if $\omega = \omega_c$, and are absent, if conditions (5) are not met. There are no real solutions for equation $P_2(k) = 0$. In contrast, Eqs. $P_3(k) = 0$ and $P_4(k) = 0$ always possess unique real solutions, such as k_3 and k_4, respectively, with $k_3 > k_4$. The properties of k_s $(s = 1, ..., 4)$ and waves corresponding to them are essentially similar to the case of a homogeneous fluid of finite depth discussed in [5].

In a limiting case of an infinitely great depth of the upper layer $H \to \infty$, the k_s solutions are equal to

$$k_{1,2} = \frac{\bar{g}}{2U^2}(1 - 2\tau \pm \sqrt{1-4\tau}), \quad k_{3,4} = \frac{\bar{g}}{2U^2}(1 + 2\tau \pm \sqrt{1+4\tau})$$

With the numerical method mentioned, there have been determined the hydrodynamic loads on submerged circular and elliptical cylinders. The steady loads (wave resistance, lift and moment) on a submerged cylinder advancing in two-layer fluid bounded either by a rigid lid or by a free surface can be found in [6]. The solutions of radiation and diffraction problems without forward speed for two types of boundary conditions are presented in [7]. The version of the Haskind-Newman relation is derived for two-layer fluid.

In a case when two-layer fluid is bounded by a free surface, both surface and internal wave modes exist. An interesting peculiarity of the diffraction problem for stratified fluid is that when the given mode wave incidents on a body it scatters not only into itself but also into all the other modes. This is one of the mechanisms of energy redistribution due to wave motions, in particular, that of the surface wave energy transfer to depth.

In [6, 7] the comparison is made between numerical solutions and the approximate analytical solutions based on the use of the Kochin function, valid for deeply submerged elliptical cylinders under the interface. In a steady problem, the approximate solution is obtained for a wave resistance, and for all the characteristics of radiation and diffraction loads at $U = 0$.

The approximate solutions are obtained for forward-speed radiation and diffraction problems for deeply submerged elliptical cylinder. The diagonal damping coefficients and exciting forces are calculated in a manner like that in [5]. All the components of radiation and diffraction loads are compared for different cases when a cylinder moves under a free surface in homogeneous fluid or under an interface between two fluids bounded by a rigid lid or unbounded in vertical direction. The fluid stratification is shown to affect significantly the hydrodynamic characteristics of a submerged body over certain ranges of the body movement velocities and incident wave frequencies.

References

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DISCUSSION

Eatock Taylor R.: Is the fact that $\Lambda_{31} = \Lambda_{13}$ etc. in your numerical results a consequence of a very accurate discretisation, or does it follow automatically from the finite element formulation you have used?

Sturova I.V.: I have presented the numerical results which related to relatively small velocity of body U. In this case the relations of Timman-Newman take place. But for larger velocity U these relations are not valid and the numerical results confirm this.

Grue J.: Consider an oscillating cylinder at zero forward speed. How large can the amplitudes of the internal waves be compared to the amplitudes of the free surface waves?

Sturova I.V.: For small frequencies of the cylinder oscillations and at zero forward speed, the amplitude of the internal waves can be much greater than that of the surface waves. The same is valid also for a non-zero forward speed of the cylinder.