Study of the second-order sea-keeping problem for submerged bodies

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Introduction

The present work deals with the three-dimensional second-order radiation problem for submerged bodies : there is no incoming wave and the motion of the body is time-periodic. this includes the non-homogeneous free-surface condition. In the first section, we introduce the problem and then, in the second section, propose a method of solution by the "Limiting Absorption Principle", and present first numerical results.

1 Equations of the problem

We assume water to be an ideal and incompressible fluid and its motion to be irrotational. This implies the existence of a velocity potential φ, satisfying the usual equations of hydrodynamics. We suppose that φ can be developped in powers of a small parameter ε which measures the amplitude of the motion :

$$\varphi = \varepsilon\varphi^1 + \varepsilon^2\varphi^2 + \mathcal{O}\left(\varepsilon^3\right).$$

We obtain the first-order problem P_t^1, whose solution is φ^1 and the second-order system of equations P_t^2, verified by φ^2 :

$$P_t^i \begin{cases} (a) & \Delta\varphi^i = 0 & \text{in } \Omega_0, \\ (b) & \partial_{tt}^2\varphi^i + g\,\partial_z\varphi^i = Q^i & \text{on } z = 0, \\ (c) & \partial_n\varphi^i = F^i & \text{on } \Sigma, \\ (d) & \lim \partial_z\varphi^i = 0 & \text{at } z \to -\infty, \end{cases}$$

where

$$\begin{cases} Q^1(x,y;t) = 0, \\ Q^2(x,y;t) = -2\nabla\varphi^1 \cdot \partial_t\left(\nabla\varphi^1\right) + \dfrac{1}{g}\,\partial_t\varphi^1\,\partial_z\left(\partial_{tt}^2\varphi^1 + g\,\partial_z\varphi^1\right), \\ F^i(x,y,z;t) = f^i(x,y,z)\,e^{-i\omega t}\ ;\ f^i \text{ belongs to } L^2(\Sigma), \end{cases}$$

and Ω_0 is the fluid domain, Σ is the body boundary, and $\vec{n}$ is the external normal. We suppose that the body oscillates with pulsation ω ; we may write :

$$\varphi^1(x,y,z;t) = \mathfrak{Re}\left(\Phi_R^1(x,y,z)\,e^{-i\omega t}\right).$$

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Noting $\nu = \omega^2/g$, the first-order harmonic problem P^1 whose solution is Φ_R^1 can be written :

$$P^1 \begin{cases} (a) \quad \Delta\Phi_R^1 = 0 & \text{in } \Omega_0, \\ (b) \quad \partial_z \Phi_R^1 - \nu\, \Phi_R^1 = 0 & \text{on } z=0, \\ (c) \quad \partial_n \Phi_R^1 = f^1 & \text{on } \Sigma, \\ (d) \quad \lim \partial_z \Phi_R^1 = 0 & \text{at } z \to -\infty. \end{cases}$$

The solution satisfying the outgoing radiation condition

$$\lim_{r\to+\infty} \int_{z=-\infty}^{0} \left(\int_{C(r)} \left| \frac{\partial \Phi_R^1}{\partial \xi} - i\nu\, \Phi_R^1 \right|^2 d\xi \right) dz \quad = \quad 0, \tag{1}$$

(where $C(r)$ is a circle of radius r, so that the infinite vertical cylinder of section $C(r)$ contains the body) can be obtained by the "Limiting Absorption Principle".
If we replace φ^1 by its expression in condition (b) of the second-order transient problem, we obtain that $Q^2(x,y;t) = \Re e\,(Q_{2\omega}^2(x,y)\, e^{-2i\omega t} + Q_0^2(x,y))$, and consequently :

$$\varphi^2 = \Re e \left(\Phi_{2\omega}^2\, e^{-2i\omega t} + \Phi_0^2 \right).$$

P^2 can be decomposed into two independent problems : the first one, denoted P_0^2 of pulsation 0 whose solution is Φ_0^2 and the second, called $P_{2\omega}^2$ of pulsation 2ω a solution of which is $\Phi_{2\omega}^2$. In this paper, we just deal with $P_{2\omega}^2$ because the free-surface condition of P_0^2 is a Neumann condition with a second member in $L^1(\mathbb{R}^2)$, which should be easier to solve.

$$P_{2\omega}^2 \begin{cases} (a) \quad \Delta\Phi_{2\omega}^2 = 0 & \text{in } \Omega_0, \\ (b) \quad \partial_z \Phi_{2\omega}^2 - 4\nu\, \Phi_{2\omega}^2 = \dfrac{1}{g}\, Q_{2\omega}^2 & \text{on } z=0, \\ (c) \quad \partial_n \Phi_{2\omega}^2 = f^2 & \text{on } \Sigma, \\ (d) \quad \lim\ \partial_z \Phi_{2\omega}^2 = 0 & \text{at } z \to -\infty, \end{cases}$$

where

$$Q_{2\omega}^2 = i\omega \left(\left[\nabla \Phi^1 \right]^2 - \frac{1}{2} \Phi^1 \left[\partial_{zz}^2 \Phi^1 - \nu\, \partial_z \Phi^1 \right] \right)_{|z=0}.$$

The system of equations $(a)-(b)-(c)-(d)$ does not have a unique solution. As for the first-order problem, we will use the "Limiting Absorption Principle" to choose one of them.

2 A method of resolution for $P_{2\omega}^2$

As $\Phi_{2\omega}^2$ depends on Φ^1 via condition (b), we must first obtain the solution of P^1. It is a classical problem. The second-order problem $P_{2\omega}^2$ is more difficult because the free surface condition is non-homogeneous and has an unbounded support. In the following, we propose a way to solve $P_{2\omega}^2$.

2.1 Decomposition of $P_{2\omega}^2$

P.D. Sclavounos (see [5]) splits the solution of a problem similar to $P_{2\omega}^2$ into the sum of a particular solution Φ_P and an homogeneous solution Φ_H. Φ_P is harmonic in the fluid domain and satisfies the non-homogeneous free-surface condition. He writes Φ_P as :

$$\Phi_P(\mathbf{x}) = \int_{\text{body}} \sigma(\xi)\, \mathcal{R}(\mathbf{x},\xi)\, d\xi,$$

where $\sigma(\xi)$ is the operator involved in the integral representation of the first-order potential, and $\mathcal{R}(\mathbf{x},\xi)$ is the second-order Green function (see section 3 in [5]). Concerning Φ_H, it is harmonic in Ω_0, satisfies the homogeneous free-surface condition and the condition $\partial\Phi_H/\partial n = -\partial\Phi_P/\partial n$ on the body boundary. He needs the second-order Green function (to calculate Φ_P) and to take the limit **on the body** of the gradient of the integral defining Φ_P, to know Φ_H. We propose a method which avoids this. We choose the same decomposition as M. Verrière (see [6]) for the three-dimensional transient problem of tsunamis, employed latter by O. Mechiche Alami for the case of acoustics (see [4]) and then by A. Friis, J. Grue and E. Palm (see [2]) for the two-dimensional sea-keeping problem with a bichromatic swell. This method introduces an auxiliary problem $P^{21}_{2\omega}$, which consists in ignoring the body in the fluid domain :

$$P^{21}_{2\omega}\quad\begin{cases}(a) & \Delta\Phi^{21}_{2\omega}=0 & z<0,\\ (b) & \partial_z\Phi^{21}_{2\omega}-4\nu\,\Phi^{21}_{2\omega}=\dfrac{1}{g}\,Q^2_{2\omega} & \text{on } z=0,\\ (c) & \lim\partial_z\Phi^{21}_{2\omega}=0 & \text{at } z\to-\infty.\end{cases}$$

We substract the solution of $P^{21}_{2\omega}$ from the solution of $P^2_{2\omega}$:

$$\Phi^2_{2\omega}-\Phi^{21}_{2\omega}=\Phi^{22}_{2\omega},$$

where $\Phi^{22}_{2\omega}$ satisfies the following equations :

$$P^{22}_{2\omega}\quad\begin{cases}(a) & \Delta\Phi^{22}_{2\omega}=0 & \text{in } \Omega_0,\\ (b) & \partial_z\Phi^{22}_{2\omega}-4\nu\,\Phi^{22}_{2\omega}=0 & \text{on } z=0,\\ (c) & \partial_n\Phi^{22}_{2\omega}=f^{(2)}-\partial_n\Phi^{21}_{2\omega} & \text{on } \Sigma,\\ (d) & \lim\partial_z\Phi^{22}_{2\omega}=0 & \text{at } z\to-\infty.\end{cases}$$

To find $\Phi^2_{2\omega}$ amounts to calculating the solutions of problems $P^{21}_{2\omega}$ and $P^{22}_{2\omega}$.

2.2 A resolution method for $P^{21}_{2\omega}$ and $P^{22}_{2\omega}$

To find a solution of $P^{21}_{2\omega}$, we apply the horizontal Fourier transform $\mathcal{F}$ on it (that is licit because we ignore the body), and by the "Limiting Absorption Principle", we find the solution :

$$\Psi(u,v,z)=\mathcal{F}\left(Q^2_{2\omega}\right)(u,v,z)\,\mathcal{F}(G_0)(u,v,z),$$

where $G_0(x,y,z)$ is the Green function of the first-order problem for a pin-point source located on the free-surface. Then, $\mathcal{F}^{-1}(\Psi)$ is a particular solution of $P^{21}_{2\omega}$. In practice, to evaluate $\Phi^{22}_{2\omega}$, we just have to calculate $\mathcal{F}\left(Q^2_{2\omega}\right)$ because $\mathcal{F}(G_0)(u,v,z)$ is known analytically, and then apply $\mathcal{F}^{-1}$.
Denoting $r=\sqrt{x^2+y^2}$, the asymptotic expansion of $Q^2_{2\omega}$ is :

$$Q^2_{2\omega}=\frac{e^{ikr}}{r^2}\,g(\theta)+\mathcal{O}\left(\frac{1}{r^3}\right),$$

where $g(\theta)$ is 2π-periodic (see [1]). This shows that $Q^2_{2\omega}$ does not belong to $L^1(\mathbb{R}^2)$. This difficulty does not appear in [2] because they study the two-dimensional case. We split $Q^2_{2\omega}$ into a function Q_1 which is in $L^1(\mathbb{R}^2)$, and Q_2 which is not. These two functions derive from the first term of the asymptotic expansion of $Q^2_{2\omega}$. $\mathcal{F}(Q_1)$ can be computed by the Fast Fourier Transform algorithm whereas $\mathcal{F}(Q_2)$ is calculated by a mixing analytical and numerical methods.
We now consider $P^{22}_{2\omega}$. The second member of the condition on the body is easy to calculate because it amounts to differentiate a Fourier transform. We can see that $P^{22}_{2\omega}$ has the same form as P^1. This is why, using the "Limiting Absorption Principle" for the problem $P^{22}_{2\omega}$, we find a solution verifying the radiation condition (1). $P^{22}_{2\omega}$ can then be solved by the same way as P^1.

2.3 First numerical results

We use the method in the case of a sphere of radius $\lambda/2$ and whose center is located at z=$3\lambda/2$, where λ is the wavelength. We show the modulus on $z=0$ of Φ_R^1 (fig. 1) thanks to MELINA* code and the one of $Q_{2\omega}^2$ (fig. 2).

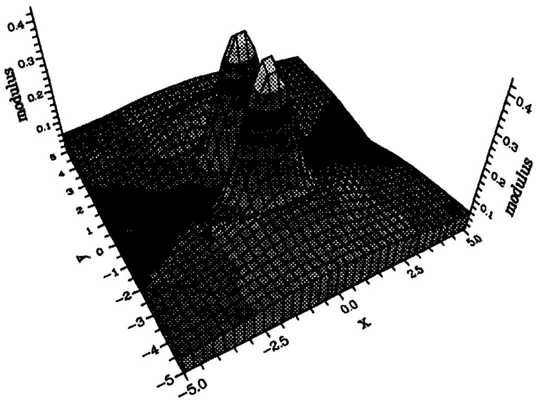

fig. 1

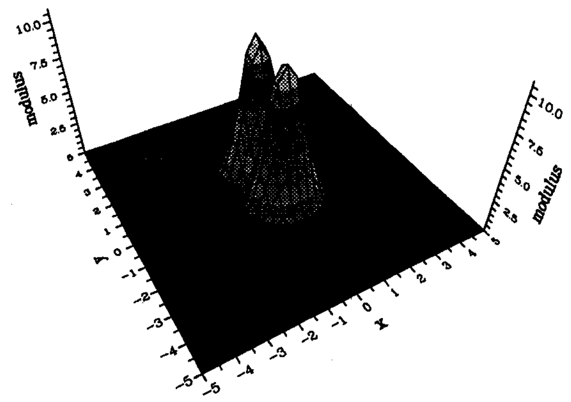

fig. 2

Conclusion

At the present time, we carry on the resolution of $\mathcal{F}(Q_{2\omega}^2)$. We have to apply $\mathcal{F}^{-1}$ which is not so easy because $\mathcal{F}(G_0)$ contains a finite part. Then, we are able to solve the radiation problem because the resolution of $P_{2\omega}^{22}$ is classical. We intend to find a relation between Φ_P obtained by "Limiting Amplitude Principle" and our solution of $P_{2\omega}^{21}$ obtained by "Limiting Absorption Principle". To solve the second-order see-keeping problem, we will have to solve the diffraction problem ; the method we propose could be applied on it and the difficulty will be that the second member of the free-surface is decreasing very slowly.

References

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[4] **O. Mechiche Alami** : *"Influence de la houle sur le rayonnement acoustique à très basse fréquence d'un corps sous-marin"* - Rapport de Recherche ENSTA n° 259 - February 1992.

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*Finite Elements code developped in "Simulation et Modélisation des Phénomènes de Propagation" which can solve problems governed by Partial Differential Equations in dimension 2 or 3.

DISCUSSION

Newman, J. N.: Can your method be extended to a floating body?

Bellier, J. L. & Champy, I.: I think that for the second order problem concerning a floating body, the difficulty is not in the method for solving the problem but in the equations themselves. For the first-order problem, concerning a floating body, we can give a sense to the equations and establish existence and uniqueness theorems. The situation is more difficult for the second-order equations.