Interaction of water waves with floating and submerged circular plates

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1 Introduction

The interaction of waves with circular plates in deep water is examined. Linear wave theory is employed and the three-dimensional problems are expressed in their integral formulation. The radiation and scattering problems for a floating disc are solved by the Boundary Element Method making use of the fact that the Green's function satisfies the free-surface condition on the plate.

When the plate is submerged, the problem can be formulated by means of a hypersingular integral equation. For two-dimensional problems a method of solution was adopted by Parsons & Martin [1] where the discontinuity in the potential across the plate was expanded in terms of Chebyshev polynomials of the second kind. This work has been extended and further applications are given in [2] and [3]. We generalise the method for our three-dimensional problems: now the double hypersingular integral over the circular plate is evaluated analytically by choosing a suitable expansion of the potential discontinuity in terms of appropriate orthogonal functions. Physical quantities are obtained numerically; in particular the occurrence of negative added mass is noticed.

2 Formulation and Solutions

Cartesian coordinates (x, y, z) are chosen with the origin in the mean free surface; the water occupies the region where $z < 0$. Linear water wave theory is employed and, under its usual conditions, the time-harmonic motion is expressed by

$$\Phi(x, y, z, t) = \mathrm{Re}(\phi(x, y, z)\, \mathrm{e}^{-i\omega t}).$$

A plane incident wave is represented by

$$\phi_{\mathrm{inc}} = \frac{g\,A}{\omega}\, \mathrm{e}^{K(z+ix)},$$

where A is the amplitude, ω is the frequency and K is the wavenumber satisfying the usual dispersion relation for infinite depth: $K = \omega^2/g$.

The potential ϕ is now written as

$$\phi = \phi_{\mathrm{inc}} + \phi_{\mathrm{sc}}$$

with $\phi_{\rm sc}$ describing the scattered wave. $\phi_{\rm sc}$ satisfies Laplace's equation in the water, the free-surface condition

$$K\phi_{\rm sc} - \frac{\partial \phi_{\rm sc}}{\partial z} = 0,$$

the radiation condition and the condition of no flow through the disc,

$$\frac{\partial \phi_{\rm sc}}{\partial n} = -\frac{\partial \phi_{\rm inc}}{\partial n}, \tag{1}$$

where n is a unit normal on the disc.

Integral equations are derived using the deep-water three-dimensional Green's function, G. This satisfies Laplace's equation in the water, the free surface condition and the radiation condition; it can be expressed in the form

$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{R} + \int_0^\infty \frac{\mu + K}{\mu - K}\, e^{(z+\zeta)\mu}\, J_0(\mu r)\, d\mu$$

where $r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}$ and $R = [r^2 + (z-\zeta)^2]^{1/2}$.

2.1 Solution for a floating disc

The disc, D is on (x,y)-plane with its centre at the origin of the system. Applying Green's theorem to $\phi_{\rm sc}$ and G and using (1) together with the fact the G satisfies the free surface condition on the disc, we find

$$-4\pi\phi_{\rm sc}(\boldsymbol{p}) = \int_D G(\boldsymbol{p},\boldsymbol{q}) \left(K\phi_{\rm sc}(\boldsymbol{q}) + \frac{\partial \phi_{\rm inc}}{\partial n}(\boldsymbol{q}) \right)\, dD_{\boldsymbol{q}},$$

where the normal has been taken upwards, $\boldsymbol{p}$ is any point in the water or on D, and $\boldsymbol{q}$ is on D. This equation is solved by the Boundary Element Method, using a piecewise-constant approximation. After choosing a set of N panels S_j, $j = 1, \ldots N$ over the disc, the approximate value of the potential on S_j, $\phi_{\rm sc}^j$, solves the linear system

$$(M + I)\, \phi_{\rm sc}^i = -M\, \frac{\partial \phi_{\rm inc}}{\partial n}^i,$$

with

$$M_{ij} = \frac{1}{4\pi} \int_{S_j} G^i\, dS_j, \tag{2}$$

where the superscript i on $\partial\phi_{\rm inc}/\partial n$ and G indicate the evaluation of these functions at a chosen point in panel S_i. Some integrations in (2) are carried out analytically and while about one hundred panels seemed to be sufficient to compute the solution for the scattering problem, 36 panels showed good results for radiation problems. Physical quantities, such as the differential scattering cross-section and damping coefficients (for the radiation problems), have been calculated.

2.2 Solution for a submerged disc

Consider a submerged disc with its centre on the z-axis. The disc radius is taken as 1, without losing any generality. The discontinuity in the potential across the disc, $[\phi]$ is given as the solution of the following hypersingular integral equation

$$-\frac{\partial}{\partial n_p}\phi_{\text{inc}}(\boldsymbol{p}) = \frac{1}{4\pi} \oint_D [\phi] \, \frac{\partial^2}{\partial n_p \partial n_q} \, G(\boldsymbol{p}, \boldsymbol{q}) \, dS_q$$

subject to condition $[\phi] = 0$ on the edge of D. In fact, this equation is valid for a thin plate of any shape (it need not be plane). For a circular plate, the kernel is given by

$$\frac{\partial^2 G}{\partial n_p \partial n_q} = \frac{1}{R^3} + \mathcal{K}(\boldsymbol{p}, \boldsymbol{q}),$$

where $R = |\boldsymbol{p} - \boldsymbol{q}|$ and $\mathcal{K}$ is regular and known. To solve this equation numerically, let (s, α) be plane polar coordinates on the disc. Then, expand $[\phi]$ using Fourier series in the angular variable

$$[\phi(s, \alpha)] = \sum_{k=0}^{\infty} c_k(s) \, \cos k\alpha. \tag{3}$$

It is known that if the coefficients $c_k(s)$ are expanded in terms of associated Legendre functions as

$$c_k(r) = \sum_{j=0}^{\infty} w_j^k \, \mathrm{P}_{k+2j+1}^k(\sqrt{1-r^2})/\mathrm{P}_{k+2j+1}^{k+1}(0), \tag{4}$$

where w_j^k are constant coefficients, then

$$\oint_D \frac{1}{R^3} \, [\phi] \, dS_q = \sum_{k=0}^{\infty} \sigma_k(r) \, \cos k\theta,$$

where

$$\sigma_k(r) = \frac{\pi}{2} \sum_{j=0}^{\infty} w_j^k \, \frac{(2j+1)!}{(2k+2j+1)!} \, \mathrm{P}_{k+2j+1}^{k+1}(0) \frac{\mathrm{P}_{k+2j+1}^k(\sqrt{1-r^2})}{\sqrt{1-r^2}}.$$

These formulae give the exact solution for potential flow past a rigid disc in an unbounded fluid. For our problem, they are convenient for the following reasons: they allow the double hypersingular integral to be evaluated analytically; and they imply that the edge condition is satisfied, because the coefficient in (4) vanishes at $r = 1$.

Substituting truncated versions of the expansions (3) and (4) into our hypersingular integral equation, a collocation scheme gives a linear system to be solved. The matrix is Vandermonde-like and is ill conditioned by its nature. Collocation at points chosen at random or on a spiral are found to give efficient collocation schemes.

The method was implemented for horizontal and inclined discs. For scattering problems, the known relation in terms of the Kochin function, H (see [4])

$$\mathrm{Im}\,(H(0)) = \frac{K^2}{4\pi\omega A} \int_0^{2\pi} |H(\theta)|^2 \, d\theta,$$

was verified. The added-mass and damping coefficients were computed for the radiation problem. The added mass presented negative values over a range of frequencies when the disc is sufficiently close to the free surface. Increasing sharp maxima in the added mass, as the frequency gets small, were noticed. These maxima are accompanied by rapid drops in the damping coefficients as frequency increases. This confirms a result of McIver & Evans [5] deduced from the Kramers–Kronig relations.

In order to understand the behaviour of the added mass for small frequencies we are currently studying the problem of two identical parallel discs in an unbounded fluid; this is equivalent to the free-surface problem for a submerged disc when $K = 0$. The aim is to obtain low-frequency asymptotics, especially when the disc is close to the free surface.

References

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[2] N. F. Parsons & P. A. Martin, *Scattering of water waves by submerged curved plates and by surface-piercing flat plates*, Appl. Ocean Res. **16**, 129–139 (1994).

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[4] J. N. Newman, *The drift force and moment on ships in waves*, J. Ship Res. **11**, 51–60 (1967).

[5] P. McIver & D. V. Evans, *The occurrence of negative added mass in free-surface problems involving submerged oscillating bodies*, Journal of Engineering Mathematics **18**, 7–22 (1984).

DISCUSSION

Yeung, R. W.: Can you comment on the type of singularity of the potential occurring at the edge of the submerged plate? Does it appear to be weaker than the case of a 2-D plate where it is known to behave like $r^{1/2}$, r being distance from the edge? Is your discretization scheme consistent with this singular behaviour?

Farina, L. & Martin, P.: The behaviour is as in 2-D: the jump in potential across the plate, $[\phi]$, behaves like $r^{1/2}$ as $r \to 0$. This square-root zero is incorporated explicitly, using (3) and (4).

Yeung, R. W.: For the submerged disc, any antisymmetric boundary condition about the y-axis (say a rolling plate or wave diffraction about the plate) would yield a net circulation on the plate. This would invalidate your potential formulation unless a vortex sheet is introduced at the edge of the plate. Please comment on this point. My remark does not suggest any negativism towards the correctness of your solution for the boundary-value problem as stated.

Farina, L. & Martin, P.: No. We do not consider a lifting flow. Perhaps it is simplest to consider a submerged spheroid with b as its smallest diameter. There is no difficulty in calculating the added-mass and damping coefficients for the spheroid (in principle, at least) when it undergoes heave or roll oscillations, say. Now let $b \to 0$; the resulting problem is that considered by us.

Kuznetsov, N. : What is the physical meaning of your homogeneous edge condition in the case of a submerged disc?

Farina, L. & Martin, P.: We have explained the problem above. We seek a bounded potential everywhere in the fluid. It can be discontinuous across the plate but not in the fluid; hence $[\phi] = 0$ at the edge.

Falnes, J.: Your results indicate a sharp drop in the added mass at a certain short interval of Ka (a is the disc radius) which means that there is a rather sudden increase in the stored potential energy (as compared to the kinetic energy) [5]. Could the corresponding wave elevation above the disc be approximately described by a Bessel function, and could this point of view be of use to quantitatively explain the occurrence of the abrupt change in added mass?

Farina, L. & Martin, P.: Thank you for your encouragement. We are actively seeking such an approximation, motivated by the paper of Miles entitled *Resonant amplification of gravity waves over a circular sill* (*JFM* **167** (1986) 169-79). His problem (bottom-mounted, vertical, truncated, circular cylinder) has three dimensionless parameters; the problem of a submerged horizontal disc in deep water is attractive for further study, as it depends on only two parameters.

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Peregrine, D. H.: Have you investigated trapped modes in connection with the strong variation of added mass?

Farina, L. & Martin, P.: Not yet. We do not expect to find any true trapped modes; if they do exist, they would provide a counter-example to the long-sought general uniqueness theorem for submerged 3-D bodies. However, we do expect to find 'leaky modes', as studied by Miles (see reply to Falnes) following Longuet-Higgins (1967).