A time-stepping model for two-dimensional nonlinear interfacial waves

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Introduction

Internal waves may propagate in fluids when density gradients due to variations in temperature or salinity are present. Knowledge of such waves, and the local currents introduced, may be relevant in many practical examples, e.g. for dimensioning of subwater bridges, which has been proposed, and for dimensioning of mooring lines or tendons of offshore structures. Transportation of pollution or mixing between sublayers in the sea are other examples.

Motivated by such problems, we consider waves which may propagate along an interface separating two fluids of different, but constant densities. The model is fully nonlinear, which is relevant for many practical examples. Furthermore, we consider a time-stepping model, which makes it possible to study generation and deformation of interfacial waves. The effect on interfacial waves by a submerged body or bottom topography is also of interest.

Propagation of unidirectional internal waves has drawn considerable interest in the literature. Some references are Long (1956), Benjamin (1966), Ono (1975), Joseph (1977), Apel et al (1985), Pullin and Grimshaw (1988). The theoretical works are mainly divided into two categories. The first category includes weakly nonlinear theory; KdV-theory, Benjamin-Ono theory, 'finite-depth theory', which may be applied to study propagation and deformation of internal waves. The second category includes strongly nonlinear wave solutions of permanent form. Our contribution will apply to both categories, and will provide a tool for analyzing fully nonlinear transient interfacial waves.

Mathematical formulation

Two-dimensional motion of horizontal fluid layers of densities $\rho_1 > \rho_2$ is considered. The bottom of the lower fluid is located at $y = -h_1$, and the top of the upper fluid at $y = h_2$. We introduce for simplicity the rigid lid approximation instead of using a free upper surface. The method is however easily extended to a free surface on top.

The fluids are assumed perfect, and the flow in each layer to be irrotational. The continuity equation is then the Laplace equation for the velocity potential in each layer. Complex coordinates $z = x + iy$ are introduced, with the real axis being horizontal in the position of the undisturbed interface separating the two fluids, and the imaginary axis being vertical with gravity acting in the negative direction.

The complex velocity, $q_1(z) = u_1 - iv_1$ for lower fluid and $q_2(z) = u_2 - iv_2$ for the upper fluid, are then analytic functions of z. By use of Cauchy's integral theorem we obtain two complex integral equations for the fluid velocities which read

$$\begin{aligned} \pi i q_1(z') &= PV \int_F \frac{q_1(z)dz}{z' - z} + \int_F \frac{q_1(z)^* dz^*}{z^* - 2ih_1 - z'} \\ -\pi i q_2(z') &= PV \int_F \frac{q_2(z)dz}{z' - z} + \int_F \frac{q_2(z)^* dz^*}{z^* + 2ih_2 - z'} \end{aligned} \quad (1)$$

where PV denotes principal value. The integral equations are discretized and solved numerically with the additional requirement of continuous normal velocities at the interface.

Knowing the velocities and the profile of the wave we are able to calculate the time development of the wave. We follow Lagrangian particles in time, these will flow with the fluid velocity that are known from the solution of the integral equations. In addition the condition of continuity of pressure at the interface gives the time derivative of the potential difference, $\rho_1\phi_1 - \rho_2\phi_2$. Knowing these derivatives it is possible to integrate to obtain the values at a later timestep, then the procedure repeats. We also apply a higher order time-integrator, along the lines introduced for free surface flow by Tanaka et al (1987).

Solitary waves interacting with a semicircular obstacle

In many physical situations there is a thin upper layer above a thicker lower layer, such as in a fjord with a layer of fresh water above salt water. We want to model such waves and study their interaction with bodies or bottom topography. As an example we study a solitary wave deflected into the lower layer, running over an obstacle at the bottom. We choose the obstacle to be a semicircle. The amplitude relative to the upper layer thickness is $a/h_2 = 0.7$. The ratio of thicknesses and the densities are $h_1/h_2 = 4$ and $\rho_2/\rho_1 = 0.9$ and the radius of the obstacle $R/h_2 = 2$. The solitary wave solution used as the initial condition in this problem is obtained by the equations (1) together with dynamic and kinematic boundary conditions.

Figure 1 shows a timeseries of the wave, increasing time is upwards. The solitary wave deforms when it is passing the obstacle but afterwards it obtains nearly its initial profile and very little energy is reflected. We will study how the obstacle alter the velocity field of the wave. Figure 2 shows the horizontal velocity as a function of the vertical coordinate at the horizontal position of the crest. It shows a comparison between the velocity induced by the undisturbed solitary wave and when the crest is passing the center of the obstacle.

The velocities in the lower layer are significantly altered due to the obstacle and the shear velocity jump is considerable increased compared to the undisturbed solitary wave. Due to the increased shear velocity the flow becomes more sensitive to Kelvin-Helmholtz instability. In this idealized model without viscosity and surface tension this instability will always be present at a sufficient small length scale, but with not to densed discretization and smoothing of the shortest perturbations it will not be present in the numerical scheme.

Increasing the amplitude of the solitary wave further will lead to an increased jump in the shear velocities at the interface. Thus for sufficient large amplitudes the Kelvin-Helmholtz instability may be the major mechanism in deforming the wave such that it will not be reasonable to artificially suppress this effect. In such cases it will be necessary to model the real physical mechanisms like viscosity and diffusion between the layers to describe the main evolution of the wave passing the obstacle.

Comparison with weakly non-linear theory and experiments

To verify the validity of using inviscid, but fully non-linear theory, we may compare the calculated solitary wave solutions to existing experimental results. Koop and Butler (1981) compared experimentally solitary waves to weakly non-linear theory. Their experiments were performed with freon and water, $\rho_2/\rho_1 = 0.633$. They compared a typically half breadth L of the solitary waves to the amplitude a. L is defined by letting $2La$ be the volume of the solitary wave. Figure 3 shows some of their results with $h_1 = 1.34$ cm and $h_2 = 6.95$ cm, the measured results are plotted with error-bars to indicate the scattering of the results. Together with these results the breadth of the calculated fully non-linear solitons and the weakly non-linear results are also shown. The figure shows very good agreement between calculated fully non-linear solitons and experiments. It also clearly shows the shortcoming of the weakly non-linear theory.

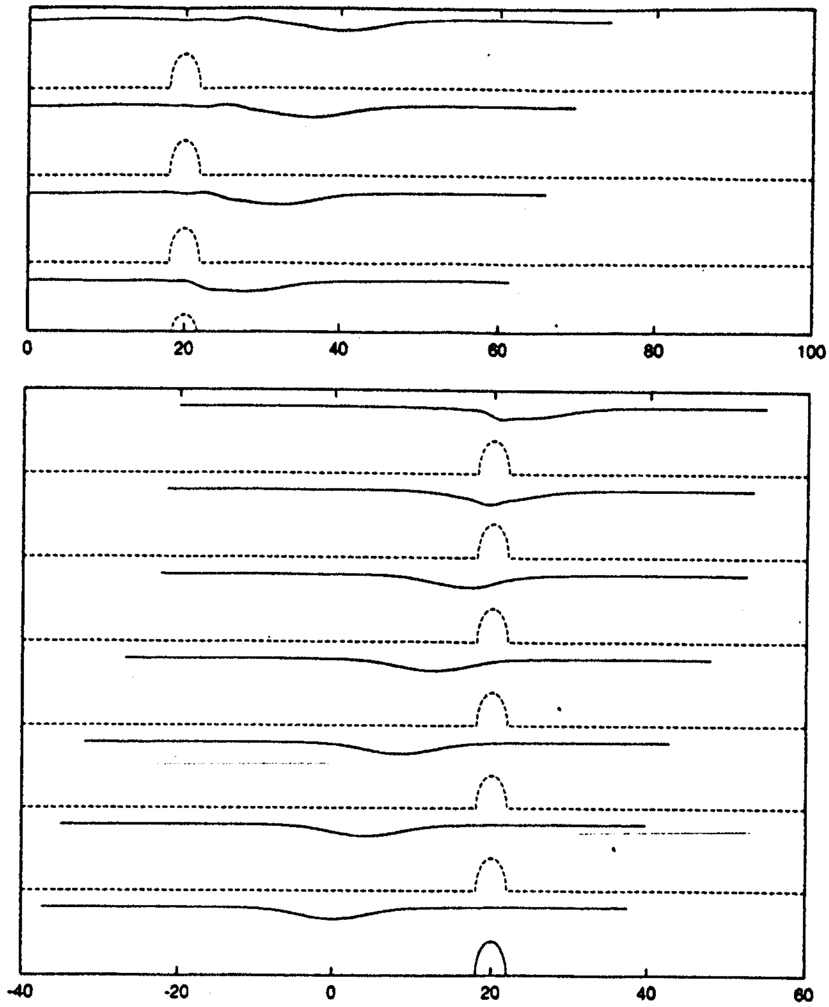

Figure 1: Timeseries of a solitary wave running over an obstacle. Increasing time is upwards. Vertical axis is stretched to double length.

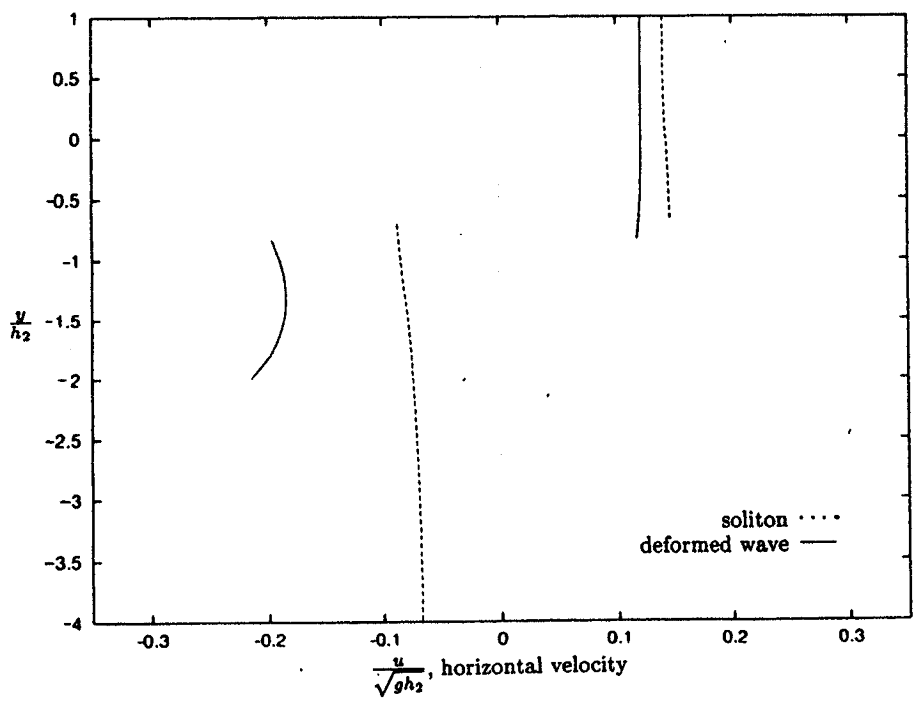

Figure 2: The horizontal velocity at the crest. Comparison between an undisturbed solitary wave and the wave when the crest is above the obstacle.

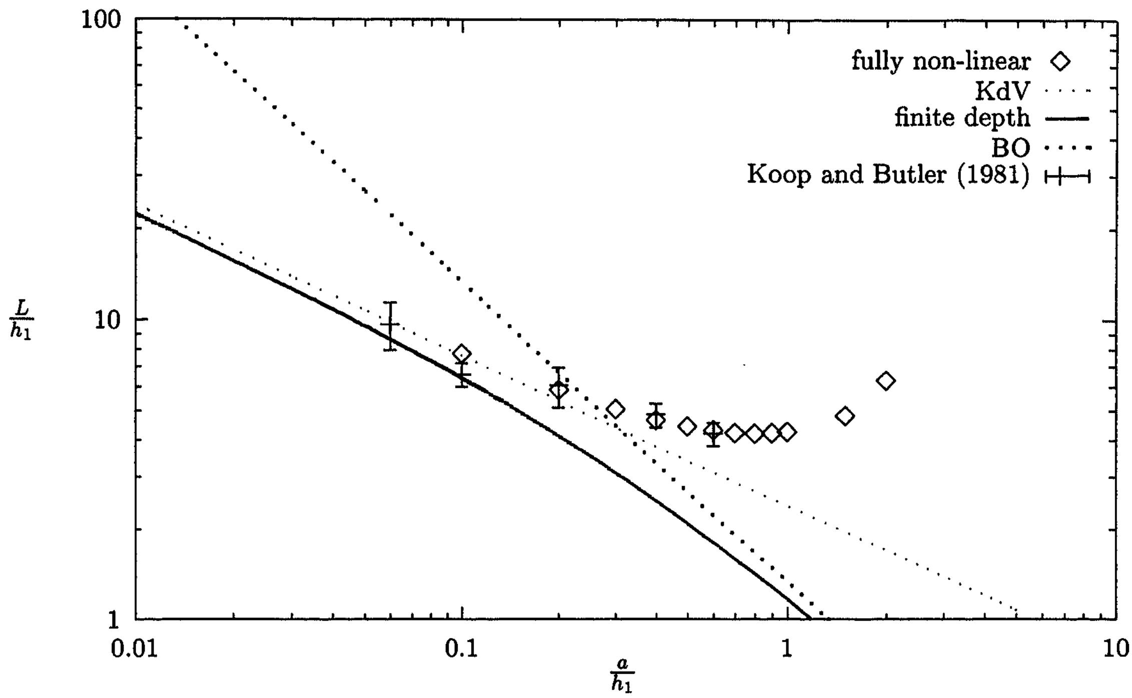

Figure 3: Solitary wave solutions from fully non-linear theory compared with weakly non-linear theory and experiments by Koop and Butler (1981). L is the half breadth and a the amplitude.

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DISCUSSION

Ma, Q. W. : Fully nonlinear analysis has attracted much attention recently. In many cases it is used for the wave problems on the interface between air and water. This paper presents an example for inner wave analysis. I am wondering if you truncated the fluid domain when you solved the equations for velocities. If yes, what kind of conditions was imposed on the truncated boundary?

Friis, A., Rusås, P-O., Grue, J. & Palm, E.: A truncated domain is applied when solving the equations since the velocities induced by the solitary wave approach zero sufficiently fast for $|x| \rightarrow \infty$. The effect of truncation is checked by increasing the length of the computational domain and the results presented are (to graphical accuracy) not affected by the truncation.

Clément, A.: You are using a "solid wall" free surface condition for the upper fluid, probably for practical reasons (may be using the mirror image of the problem with regard to this surface). Would it be far more difficult to release this constraint and use an actual free surface condition? Do you intend to do so? Because I believe that it could possibly have a significant influence in the flow features when the solitary wave is in the vicinity of the obstacle.

Friis, A., Rusås, P-O., Grue, J. & Palm, E.: It is not difficult to include a free surface on top of the upper layer as the treatment of the free surface and the interface is closely related. It will however considerably increase the computational cost as the number of unknowns is approximately doubled. We are now working with including the free surface to decide to which extent it affects the flow.