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A Nonlinear Simulation Method of 3-D Body Motions in Waves
Formulation with the Acceleration Potential

Katsuji TANIZAWA

Ship Research Institute, Shinkawa, Mitaka, Tokyo, Japan

1 Introduction

A full-nonlinear method to simulate three dimensional transient motions of floating bodies in waves will be presented. This is a time domain method to simulate Euler's equation of ideal fluid motion coupled with the equation of solid body motions. Introducing L.Prandtl's nonlinear acceleration potential, which spacial derivative gives the acceleration of fluid particle, Euler's differential equation of the ideal fluid motion is converted to the integral equation of the acceleration potential. The boundary condition of the acceleration potential on the body surface is systematically derived from the kinematic relation between the acceleration of the solid body and the accerelation of the fluid particle on the body surface. Since this kinematic boundary condition is a function of the body acceleration, the boundary values on the floating body can not be evaluated explicitly. To overcome this point, the unknown acceleration of the free floating body is eliminated by substituting the equation of body motion into kinematic condition, then implicit body surface boundary condition is derived. This is the kinematic and dynamic condition which guarantees dynamic equilibrium of forces between ideal fluid and the solid body at any instance. With the free-surface boundary condition of the accerelation potential, the boundary value problem for the accerelation field is formulated. A formulation for the numerical method is also given.

2 Euler's equation of ideal fluid and acceleration potential

First of all, let us define the nonlinear acceleration potential from Euler's equation of the ideal fluid. Non-dimensional Euler's equation of the ideal fluid ($\rho = g = 1$) can be written as

$$\boldsymbol{a} = \frac{D\boldsymbol{v}}{Dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p - \nabla Z, \tag{1}$$

where $\boldsymbol{v}$ and $\boldsymbol{a}$ are velocity and acceleration vector of the fluid particle respectively. Introducing the velocity potential ϕ , equation (1) can be written as

$$\boldsymbol{a} = \frac{D\nabla\phi}{Dt} = \frac{\partial \nabla\phi}{\partial t} + (\nabla\phi \cdot \nabla)\nabla\phi = \nabla\frac{\partial \phi}{\partial t} + \nabla\left(\frac{1}{2}(\nabla\phi)^2\right) = \nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2\right). \tag{2}$$

Here, let us define the acceleration potential Φ as

$$\Phi = \frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2, \tag{3}$$

then fluid acceleration is expressed as $\boldsymbol{a} = \nabla\Phi$. This is L.Prandtl's nonlinear accerelation potential. The acceleration field described by this acceleration potential is irrotational, but dose not satisfy Laplace's equation $\nabla^2\Phi \neq 0$ because of the nonlinearity of the second term of the right side of equation (3). From equation (1),(2) and (3), the accerelation potential is written as

$$\Phi = -p - Z + const. \quad \text{(Integral constant can be set to zero.)} , \tag{4}$$

therefore physical meaning of the acceleration potential is very clear. Despite of this clearness, the acceleration potential is rarely used to solve the hydrodynamic problems. The reason seems to be that the acceleration field is not necessary solved in the framework of linear theory. But in addition to this reason, there exist two unsolved problems. These are (1) the body surface boundary condition of the acceleration potential is not clearly obtained and (2) the acceleration potential is nonlinear and dose not satisfy Laplace's equation. This study is aimed to overcome these two problems.

3 Boundary condition of the acceleration field

3.1 Acceleration of fluid particle on the body surface

In order to get the kinematic body surface boundary condition, let us first study the acceleration of fluid particle sliding on the body surface. As illustrated in Fig.1, the space fixed reference frame $O-XYZ$ and the body fixed reference frame $o-xyz$ are used. The origin o is situated at the center of gravity of the body and the frame $o-xyz$ is moving with translating velocity $\boldsymbol{v}_o$ and angular velocity $\boldsymbol{\omega}$. In Fig.1, P is a point fixed to the fluid particle sliding on the body surface. Using the positioning vectors $\boldsymbol{R}$, $\boldsymbol{R}_o$ and $\boldsymbol{r}$ illustrated in Fig.1, the position, velocity and acceleration vector of point P are expressed as

$$\boldsymbol{R} = \boldsymbol{R}_o + \boldsymbol{r} \tag{5}$$

$$\boldsymbol{v} = \dot{\boldsymbol{R}}_o + \dot{\boldsymbol{r}} = \boldsymbol{v}_o + [\boldsymbol{v}] + \boldsymbol{\omega} \times \boldsymbol{r} \tag{6}$$

$$\begin{aligned} \boldsymbol{a} &= \ddot{\boldsymbol{R}}_o + \ddot{\boldsymbol{r}} \\ &= \boldsymbol{a}_o + [\boldsymbol{a}] + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + 2\boldsymbol{\omega} \times [\boldsymbol{v}] + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}, \end{aligned} \tag{7}$$

where $[\boldsymbol{v}]$ and $[\boldsymbol{a}]$ are velocity and acceleration of point P observed from $o-xyz$ frame respectively. With these kinematic formulae of velocity and acceleration of point P, the body surface kinematic boundary condition for the acceleration field is derived in the next section.

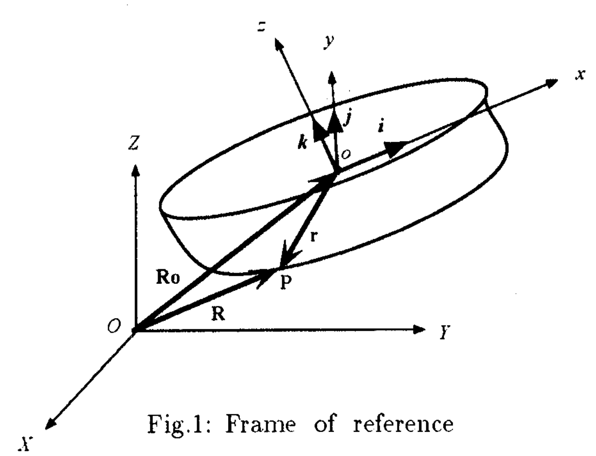

Fig.1: Frame of reference

3.2 Body surface kinematic boundary condition

Similar to the kinematic boundary condition of the velocity field, the kinematic boundary condition of the acceleration field can be expressed as scalar product of the acceleration vector of the fluid particle and the unit normal vector of the body surface at the fluid particle locates. That is

$$\frac{\partial \Phi}{\partial n} = \boldsymbol{n} \cdot \nabla \Phi = \boldsymbol{n} \cdot \boldsymbol{a}, \tag{8}$$

where $\boldsymbol{n}$ is the unit normal vector of body surface at point P. Substituting (7) into (8) gives following relation

$$\frac{\partial \Phi}{\partial n} = \boldsymbol{n} \cdot [\boldsymbol{a}] + \boldsymbol{n} \cdot (\boldsymbol{a}_o + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}) + \boldsymbol{n} \cdot \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \boldsymbol{n} \cdot 2\boldsymbol{\omega} \times [\boldsymbol{v}]. \tag{9}$$

This is the kinematic body surface boundary condition for the acceleration field. Since the fourth term of the right side includes velocity $[\boldsymbol{v}]$, this boundary condition depends on the velocity field. So, let us rewrite equation (9) with velocity potential ϕ. First, considering the equation (6), $[\boldsymbol{v}]$ can be written as

$$[\boldsymbol{v}] = \boldsymbol{v} - \boldsymbol{v}_o - \boldsymbol{\omega} \times \boldsymbol{r} = \nabla\phi - \boldsymbol{v}_o - \boldsymbol{\omega} \times \boldsymbol{r}. \tag{10}$$

Second, normal and tangential components of $[\boldsymbol{a}]$ to the body surface can be written as

$$[\boldsymbol{a}]_n = -k_n\,[\boldsymbol{v}]^2, \qquad [\boldsymbol{a}]_s = [\dot{\boldsymbol{v}}]_s, \tag{11}$$

where k_n is the normal curvature of the body surface along with the path line of P. The value of $[\boldsymbol{a}]_s$ in equation (11) is unknown, but $\boldsymbol{n}\cdot[\boldsymbol{a}]_s$ is zero because $\boldsymbol{n}$ and $[\boldsymbol{a}]_s$ are orthogonal. So, $\boldsymbol{n}\cdot[\boldsymbol{a}]$ becomes

$$\boldsymbol{n}\cdot[\boldsymbol{a}] = \boldsymbol{n}\cdot([\boldsymbol{a}]_n + [\boldsymbol{a}]_s) = \boldsymbol{n}\cdot[\boldsymbol{a}]_n = -k_n\ [\boldsymbol{v}]^2 = -k_n\ (\nabla\phi - \boldsymbol{v}_o - \boldsymbol{\omega}\times\boldsymbol{r})^2. \tag{12}$$

Finally, the kinematic boundary condition of the acceleration field is reduced to be

$$\begin{aligned}\frac{\partial\Phi}{\partial n} &= -k_n\ (\nabla\phi - \boldsymbol{v}_o - \boldsymbol{\omega}\times\boldsymbol{r})^2 + \boldsymbol{n}\cdot(\boldsymbol{a}_o + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}) \\ &\quad + \boldsymbol{n}\cdot\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) + \boldsymbol{n}\cdot 2\boldsymbol{\omega}\times(\nabla\phi - \boldsymbol{v}_o - \boldsymbol{\omega}\times\boldsymbol{r}).\end{aligned} \tag{13}$$

3.3 Euler's equation of 3-D solid body motions

The second term of the right side of Equation (13) includes the body acceleration $\boldsymbol{a}_o$ and $\dot{\boldsymbol{\omega}}$. Therefore, the body surface boundary condition can not be determined explicitly when the body acceleration is unknown. In such a case, the equation of body motions can be used to eliminate the unknown body acceleration from equation (13). The generalized Euler's equation of 3-D solid body motions is given as

$$\begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xx} & I_{xy} & I_{xz} \\ 0 & 0 & 0 & I_{yx} & I_{yy} & I_{yz} \\ 0 & 0 & 0 & I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{Bmatrix} a_{ox} \\ a_{oy} \\ a_{oz} \\ \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ (I_{zz} - I_{yy})\omega_y\omega_z - I_{xy}\omega_z\omega_x + I_{zx}\omega_x\omega_y + I_{yz}(\omega_y^2 - \omega_z^2) \\ (I_{xx} - I_{zz})\omega_z\omega_x - I_{yz}\omega_x\omega_y + I_{xy}\omega_y\omega_z + I_{zx}(\omega_z^2 - \omega_x^2) \\ (I_{yy} - I_{xx})\omega_x\omega_y - I_{zx}\omega_y\omega_z + I_{yz}\omega_z\omega_x + I_{xy}(\omega_x^2 - \omega_y^2) \end{Bmatrix} = \begin{Bmatrix} f_x \\ f_y \\ f_z \\ M_x \\ M_y \\ M_z \end{Bmatrix}, \tag{14}$$

and its vector form can be written as

$$\mathcal{M}\cdot\boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{F}, \tag{15}$$

where $\mathcal{M}$ is the generalized inertia tensor of the body, $\boldsymbol{\alpha} = (a_{ox}\boldsymbol{i} + a_{oy}\boldsymbol{j} + a_{oz}\boldsymbol{k},\ \dot{\omega}_x\boldsymbol{i} + \dot{\omega}_y\boldsymbol{j} + \dot{\omega}_z\boldsymbol{k})$ is the generalized acceleration vector of the body and $\boldsymbol{F} = (f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k},\ M_x\boldsymbol{i} + M_y\boldsymbol{j} + M_z\boldsymbol{k})$ is the generalized force acts on the body. Vector $\boldsymbol{\beta}$, which is the second term of the left side of equation (14), appears because $o - xyz$ frame is rotational frame. Now, introducing the generalized normal vector $\boldsymbol{N} = (n_x\boldsymbol{i} + n_y\boldsymbol{j} + n_z\boldsymbol{k},\ (n_yz - n_zy)\boldsymbol{i} + (n_zx - n_xz)\boldsymbol{j} + (n_xy - n_yx)\boldsymbol{k})$, the generalized hydraulic force is given as

$$\boldsymbol{F}_f = \int_{S_s} p\ \boldsymbol{N} ds = \int_{S_s} (-\Phi - Z)\ \boldsymbol{N} ds. \tag{16}$$

Here we denote the other force (thrust, gravity etc.) as $\boldsymbol{F}_g$, then total force acts on the body is written as

$$\boldsymbol{F} = \boldsymbol{F}_f + \boldsymbol{F}_g = \int_{S_s} (-\Phi - Z)\ \boldsymbol{N} ds + \boldsymbol{F}_g. \tag{17}$$

Equation (15) and (17) gives the generalized Euler's equation of 3-D body motions coupled with fluid motion.

$$\mathcal{M}\cdot\boldsymbol{\alpha} + \boldsymbol{\beta} = \int_{S_s} (-\Phi - Z)\ \boldsymbol{N} ds + \boldsymbol{F}_g. \tag{18}$$

3.4 Implicit body surface boundary condition

The unknown acceleration is included in the second term of the left side of equation (13), and the other term can be explicitly evaluated from the solution of velocity field. So, let denote the other term as q for simplicity

$$q = -k_n\ (\nabla\phi - \boldsymbol{v}_o - \boldsymbol{\omega}\times\boldsymbol{r})^2 + \boldsymbol{n}\cdot\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) + \boldsymbol{n}\cdot 2\boldsymbol{\omega}\times(\nabla\phi - \boldsymbol{v}_o - \boldsymbol{\omega}\times\boldsymbol{r}). \tag{19}$$

Next, the second term can be written much simpler with the generalized acceleration $\boldsymbol{\alpha}$ and the generalized normal vector $\boldsymbol{N}$ as

$$\boldsymbol{n}\cdot(\boldsymbol{a}_o + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}) = \boldsymbol{n}\cdot\boldsymbol{a}_o + \dot{\boldsymbol{\omega}}\cdot(\boldsymbol{n}\times\boldsymbol{r}) = \boldsymbol{N}\cdot\boldsymbol{\alpha}. \tag{20}$$

Then, equation (13) is simply written as

$$\frac{\partial \Phi}{\partial n} = \boldsymbol{N} \cdot \boldsymbol{\alpha} + q. \tag{21}$$

Eliminating the generalized acceleration from equation (18) and (21), the implicit body surface boundary condition

$$\frac{\partial \Phi}{\partial n} = \boldsymbol{N} \cdot \mathcal{M}^{-1} \cdot \int_{S_s} -\Phi \; \boldsymbol{N} ds + \boldsymbol{N} \cdot \mathcal{M}^{-1} \cdot \left\{ \int_{S_s} Z \; \boldsymbol{N} ds + \boldsymbol{F}_g - \boldsymbol{\beta} \right\} + q \tag{22}$$

is finally derived. This condition gives the relation between the acceleration potential Φ and its flux $\partial\Phi/\partial n$ on the body surface.

3.5 Free-surface boundary condition

From equation (4), following free-surface boundary condition is given.

$$\Phi_{on \; f.s.} = -Z - p_{atm} \tag{23}$$

4 A formulation for numerical method

As mentioned before, the acceleration potential Φ does not satisfy Laplace's equation. So, Φ is not adequate for numerical method like BEM. But equation (3) shows that the nonlinear part of Φ can be explicitly determined from the solution of velocity field. Therefore it is not necessary to solve the nonlinear part with Φ . Let us subtract this part from Φ and define pseudo-acceleration potential Ψ as

$$\Psi = \frac{\partial \phi}{\partial t} = \Phi - \frac{1}{2}(\nabla \phi)^2 \; . \tag{24}$$

Now, Ψ satisfies Laplace's equation. So, with given boundary conditions, boundary value problem on Ψ is easier to be solved than that on Φ . The boundary condition for the pseudo-acceleration potential Ψ is easily obtained from equation (21),(22) and (23) as follows.

- Body surface boundary condition

$$\frac{\partial \Psi}{\partial n} = \boldsymbol{N} \cdot \boldsymbol{\alpha} + q - \frac{\partial}{\partial n}\left(\frac{1}{2}(\nabla \phi)^2\right) \tag{25}$$

- Implicit body surface boundary condition

$$\begin{aligned} \frac{\partial \Psi}{\partial n} &= \boldsymbol{N} \cdot \mathcal{M}^{-1} \cdot \int_{S_s} -\Psi \; \boldsymbol{N} ds \\ &+ \boldsymbol{N} \cdot \mathcal{M}^{-1} \cdot \left\{ \int_{S_s} \left(Z - \frac{1}{2}(\nabla \phi)^2 \right) \boldsymbol{N} ds + \boldsymbol{F}_g - \boldsymbol{\beta} \right\} + q - \frac{\partial}{\partial n}\left(\frac{1}{2}(\nabla \phi)^2\right) \end{aligned} \tag{26}$$

- Free-surface boundary condition

$$\Psi_{on \; f.s.} = -Z - p_{atm} - \frac{1}{2}(\nabla \phi)^2 \tag{27}$$

5 Conclusion

1. The body surface boundary condition of nonlinear acceleration potential is systematically derived.
2. Substituting the equation of 3-D body motions into the body surface boundary condition, the implicit body surface boundary condition is derived.
3. With the free-surface boundary condition, the mathematical formulation of the boundary value problem on the acceleration potential is given.
4. For numerical method like BEM, the boundary value problem on the pseudo-acceleration potential is also given.
5. Some numerical results of full nonlinear floating body simulation in waves will be presented at the workshop. The results show that the conservation law of mass, momentum and energy are nicely satisfied.

References

1) Vinje,T. and Brevig,P.: Nonlinear Ship Motions, *Proc. of the 3rd. Int. Conf. on Num. Ship Hydro.*,(1981)
2) Tanizawa,K.: A Numerical Method for Nonlinear Simulation of 2-D Body Motions in Waves by means of B.E.M., *Journal of the Society of Narval Architects of Japan*, Vol.168,(1990)

DISCUSSION

van Dalen, E. F. G.: I recall that we have discussed the technique described here in Val de Rueil (1992) already. Shortly after, I completed my PhD thesis (1993) which includes a profound description of the combination of $\nabla^2\phi_t = 0$ in boundary integral form and the hydrodynamic equation of motion for the body. It surprises me that you have not taken notice, directly or indirectly, of the work, whereas others have (e.g. Wu, Ma and Eatock Taylor, in this workshop). For completeness, I add the reference here: "*Numerical and Theoretical studies on water waves and floating bodies*"., PhD thesis, University of Twente, Enschede, The Netherlands, 1993.

Tanizawa, K.: Thank you for your comment. I also recall the discussion in Val de Rueil. At that time we confirmed to each other that we have almost the same idea, and I informed you that I have already published a paper on the idea in 1990. Since this paper was written in Japanese, I sent you the English translation shortly after the discussion in Val de Rueil, but I didn't receive your thesis when you finished it. This is the reason why I couldn't refer to your work. But now I have received your PhD thesis after this workshop. I will willingly add your PhD thesis to my references from now on.

In the oral discussion at the workshop, you pointed out that my talk on this topic was exactly the same as your work. But, it is not the same. I think you were talking about the idea. Yes, your idea is the same as mine described in the 1990 paper. But, this time, I gave the exact formulation for 3-D full-nonlinear problems. The important points of my work are:

1) Nonlinear acceleration potential is taken into consideration.

2) The kinematic body surface boundary conditions for this nonlinear acceleration potential can be obtained by a Lagrangian derivation.

3) The nonlinear term can be shifted from the governing equation to the boundary condition for the numerical computation. As a result, nonlinear terms appear in the boundary conditions.

Of course, we can apply the same idea to construct the simultaneous equations of fluid motions and floating body motions with this nonlinear acceleration potential.

Kring, D.: Have you studied the exact origin of the numerical time derivative of the potential. From my experience, it stems only from the instantaneous component of the potential. Our experience with the time derivatives agrees with your observations.

Tanizawa, K.: I agree with your comments. The time derivative of the velocity potential should be determined from the instantaneous conditions which guarantee dynamic equilibrium of forces between fluid and body, and should not be approximated by backward finite differences. I think the boundary value problem for the nonlinear acceleration potential which I have formulated here can be a consistent answer to determine the time derivative of the velocity potential.