Three-dimensional harmonically oscillating Green function with small forward speed in finite water depth

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A harmonically oscillating Green function, which satisfies the classical linear free surface condition with small forward speed (correct to order U), radiation condition and bottom condition, is studied. This can be used as a part of the analysis of wave induced motions and loads on large volume structures.

It can be shown that the Green function ($Ge^{i\omega t}$) can be written as

$$G(\vec{x},\vec{a}) = \frac{1}{\sqrt{r^2+(z-c)^2}} + \frac{1}{\sqrt{r^2+(z+c+2h)^2}} + G_1,$$

where G_1 is written as the double integral

$$G_1 = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{e^{ikr\cos(\theta-u+\alpha)}e^{-kh}\cosh k(z+h)\cosh k(c+h)[k(1+2\tau\cos\theta)+\nu]}{k\sinh kh - [\nu+2\tau\cos\theta k - \overline{\mu}i(\nu+\tau k\cos\theta)]\cosh kh} dk d\theta. \quad (1)$$

Here $x-a = r\cos u$ and $y-b = r\sin u$, and the parameters $\tau = \frac{U\omega}{g}$ and $\nu = \frac{\omega^2}{g}$ have been introduced. The z-axis is vertical and positive upwards. (a,b,c) is the source point. $\overline{\mu} = \frac{\mu}{\omega}$ denotes the non-dimensional Rayleigh viscosity, ω the frequency of oscillation, h the water depth and α the current angle with respect to the positive x-axis. One can either start integrating the double integral in the k-direction or in the θ-direction. These two alternative ways of calculating G_1 will be discussed here.

1 Alternative 1

In the first method one integrate first in θ-direction. The procedure is based on a general idea by Knudsen [1992]. Then G_1 can be written as

$$G_1 = \frac{1}{\pi}\int_0^{\infty} A \int_0^{2\pi} \frac{B + C\cos\theta}{D + E\cos\theta} e^{ikr\cos(\theta-u+\alpha)} d\theta dk, \quad (2)$$

where

$$A = A(k) = \cosh k(z+h)\cosh k(c+h)e^{-kh}$$

$$B = B(k) = k + \nu$$

$$C = C(k) = 2\tau k$$

$$D = D(k) = k \sinh kh - \nu(1 - \overline{\mu} i)\cosh kh$$

$$E = E(k) = -2\tau k \cosh kh (1 - \frac{\overline{\mu}}{2} i).$$

By substituting $t = e^{i\theta}$ equation 2 can be written as

$$G_1 = \frac{1}{\pi i}\int_0^\infty A(k)\int_{|t|=1}\frac{1}{t}Q(k,t)e^{\frac{ikr}{2}(\gamma t + \frac{1}{\gamma t})}dtdk. \quad (3)$$

Here $\gamma = e^{i(\alpha-u)}$ and $Q(k,t) = \frac{A+B\frac{1}{2}(t+\frac{1}{t})}{D+E\frac{1}{2}(t+\frac{1}{t})}$.

Let us assume for a moment that $\tau \neq 0$. Then $E \neq 0$ and $Q(k,t)$ has two roots in the t-plane, $\rho_1 = \frac{-D-\sqrt{D^2-E^2}}{E}$ and $\rho_2 = \frac{-D+\sqrt{D^2-E^2}}{E}$. Since $\rho_1\rho_2 = 1$, they are either both on the unit circle, or one of them must be inside and the other one outside the unit circle. By denoting the root with the smallest absolute value ρ_s and the other one ρ_l, we find

$$\rho_s = \begin{cases} \rho_1 & k < \sigma_1 \\ \rho_2 & k > \sigma_1, \end{cases}$$

where σ_1 is given by the equation $\sigma_1 \tanh \sigma_1 h - \nu = -2\tau\sigma_1$

Now, we first make a partial fraction expansion of $Q(k,t)$. Next, $Q(k,t)$ is further rewritten by applying $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$. Thus,

$$Q(k,t) = \kappa - 2\kappa_1 + \kappa_1(\frac{1}{1-\frac{\rho_s}{t}} + \frac{1}{1-\frac{t}{\rho_l}}) = \kappa + \kappa_1 \sum_{n=1}^{\infty}\left[(\frac{\rho_s}{t})^n + (\frac{t}{\rho_l})^n\right], \quad (4)$$

where $\kappa_1 = \kappa_1(k) = \frac{CD-BE}{E(\rho_l E+D)}$ and $\kappa = \kappa(k) = \frac{C}{E} + \kappa_1$.

The remaining expression in equation 3 may also be rewritten in powers in t (see Whittaker and Watson [1950] page 358);

$$e^{\frac{ikr}{2}(\gamma t + \frac{1}{\gamma t})} = \sum_{n=-\infty}^{\infty}(\gamma i)^n J_n(kr)t^n. \quad (5)$$

Inserting the new expressions into equation 3 and integrating by the method of residues, we are left with the integral

$$G_1 = 2\int_0^\infty A(k)[\kappa(k)J_0(kr) + \kappa_1(k)\sum_{n=1}^{\infty}(i\rho_s)^n J_n(kr)(\gamma^n + \frac{1}{\gamma^n})]dk, \quad (6)$$

If $|z+c|$ is small, the integrand decays slowly with increasing k. To avoid difficulties regarding when to stop integrating we follow the same technique as Faltinsen and Michelsen [1974] used for zero Froude number. That is: integrate up to a certain $k = \sigma_3$, which for $k > \sigma_3$ satisfies $1 - \tanh kh < \epsilon$. σ_3 should also be chosen larger than σ_2. In the remaining double integral we integrate with respect to k, leaving an integral in θ. $G_1^{Alt.1}$ is then written

$$\begin{aligned} G_1^{Alt.1} &= -2\int_0^{\sigma_3}\frac{\cosh k(z+h)\cosh k(c+h)e^{-kh}}{\cosh kh}J_0(kr)dk \\ &\quad -2\int_0^{\sigma_1} f(k)[J_0(kr) + 2\sum_{n=1}^{\infty}(i\rho_{s_1})^n J_n(kr)\cos n(\alpha-u)]dk \\ &\quad +2\int_{\sigma_1}^{\sigma_3} f(k)[J_0(kr) + 2\sum_{n=1}^{\infty}(i\rho_{s_2})^n J_n(kr)\cos n(\alpha-u)]dk \end{aligned} \quad (7)$$

$$+\frac{1}{2\pi}\int_0^{2\pi} e^{-\beta\sigma_3}\left[\frac{1}{\beta}\frac{1+2\tau\cos\theta}{1-2\tau\cos\theta}+\frac{2\nu}{(1-2\tau\cos\theta)^2}G[\beta(\sigma_3-\frac{\nu}{1-2\tau\cos\theta})]\right]d\theta,$$

where $f(k)=\frac{k\cosh k(z+h)\cosh k(c+h)}{\cosh^2 kh\sqrt{(k\tanh kh-\nu)^2-4\tau^2k^2}}$, $\rho_{s_1}=\frac{k\tanh kh-\nu+\sqrt{(k\tanh kh-\nu)^2-4\tau^2k^2}}{2\tau k}$, $\rho_{s_2}=\frac{k\tanh kh-\nu-\sqrt{(k\tanh kh-\nu)^2-4\tau^2k^2}}{2\tau k}$, $\beta=-[z+c+ir\cos(\theta-u+\alpha)]$, $G[\beta(\sigma_3-\frac{\nu}{1-2\tau\cos\theta})]=e^{[\beta(\sigma_3-\frac{\nu}{1-2\tau\cos\theta})]}E_1[\beta(\sigma_3-\frac{\nu}{1-2\tau\cos\theta})]$ and E_1 denotes the exponential integral.

2 Alternative 2

As the next alternative we will evaluate G_1 by first integrating in k. A similar technique as John [1950] used in the zero speed case will be applied. We write G_1 as

$$G_1=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}[e^{kc}+e^{-k(2h+c)}]e^{ikr\cos(\theta-u+\alpha)}p(k,\theta)dkd\theta, \tag{8}$$

where

$$p(k,\theta)=\frac{\cosh k(z+h)[k(1+2\tau\cos\theta)+\nu]}{k\sinh kh-[\nu+2\tau k\cos\theta-\overline{\mu}i(1+\tau k\cos\theta)]\cosh kh}. \tag{9}$$

$p(k,\theta)$ has an infinite number of simple poles, c_n, in the complex plane. The two real poles are called c_{0+} and c_{0-}, respectively. The complex poles are numbered $n=1,-1,2,-2,...$ with $|n|$ growing with larger $|c_n|$. Positive n denotes the poles with positive imaginary part. c_{-n} is the complex conjugate of c_n.

Now we rewrite $p(k,\theta)$ as the sum of its principal parts and an analytic function;

$$p(k,\theta)=g(k,\theta)+\sum_{n=-\infty}^{\infty}\frac{Res_n}{k-c_n}. \tag{10}$$

Here, $g(k,\theta)$ is the analytic function and the residues satisfy $Res_n=\frac{c_n^2e^{c_nh}\cosh c_n(z+h)}{c_n^2h+\nu\cosh^2 c_nh}$. The infinite series converges like $\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi z}{h}}{n}$, which is very slow. We will take advantage of Mittag-Leffler's theorem concerning non-convergent series in order to make our series converge faster. That means, a Taylor expansion of the series around zero is withdrawn from the series. We will start out by considering the first term in the Taylor series, only. Then,

$$p(k,\theta)=g_1(k,\theta)+\sum_{n=-\infty}^{\infty}\left(\frac{Res_n}{k-c_n}+\frac{Res_n}{c_n}\right), \tag{11}$$

where $g_1(k,\theta)$ is a new analytic function. The sum now converges as $\frac{1}{n^2}$. In fact, withdrawing L terms in the Taylor series would imply a convergence like $\frac{1}{n^{L+1}}$.

From equation 9 we see that $p(k,\theta)$ is finite in the whole complex plane away from the poles. By applying Liouville's theorem we find $g_1(k,\theta)$ to be a constant, and by comparing equation 9 and equation 11 for $k=0$, we see that the constant equals -1.

By inserting the new $p(k,\theta)$ into equation 8 we find the second alternative as

$$\begin{aligned}G_1^{Alt.2} &= -\frac{1}{\sqrt{c^2+r^2}}-\frac{1}{\sqrt{(2h+c)^2+r^2}}\\ &+\frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty}Res_n\sum_{i=1}^{2}\left\{\frac{1}{c_n\beta_i}+e^{-\beta_ic_n}\int_{-\beta_ic_n}^{\infty}\frac{e^{-t}}{t}dt\right\}d\theta,\end{aligned} \tag{12}$$

where $\beta_1 = -c - ir\cos(\theta - u + \alpha)$ and $\beta_2 = 2h + c - ir\cos(\theta - u + \alpha)$. The integrals in t are simplified by applying the exponential integral.

3 Numerical calculations and verifications

The two alternative Green functions are computed and found to agree well in a broad range of parameter variations. $G_1^{Alt.1}$ is not valid when τ equals zero. But it gives good results when $\tau > 0.001$. In calculating the last term in equation 7, σ_3 has to be chosen very large. Some extra care should be taken in evaluating this integral, including the calculation of the exponential integral.

When evaluating $G_1^{Alt.2}$, 3-4 terms in the Taylor series around zero are withdrawn (see eq. 11). This will improve the convergence considerably. The convergence gets slower for smaller r/h. The accuracy in computating $G_1^{Alt.2}$ depends on what accuracy we calculate the exponential integral.

Zhao and Faltinsen [1989] solved the interaction problem between waves, current and a structure by approximating the far-field solution by a sum of multipoles inside the structure. This implies evaluating derivatives of the Green function (up to six'th order) are needed. We find that the first alternative method works well with this respect, but the second alternative is inaccurate unless both r/h and $|c/h|$ are large.

So, when higher order derivatives are needed we have to rely on the first alternative method. To verify the results we examine numerically whether G satisfies the Laplace equation and the free surface condition. Differentiation of these equations are used to verify the multipole expressions. The results are generally satisfactory, unless $|z+c|$ is small (see eq.1). In addition, $G^{Alt.1}$ and its multipoles are compared with infinite frequency results and infinite depth results. Details about the numerical calculations and verifications can be found in Bratland [1996].

References

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