Free surface flows with several stagnation points

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1 Introduction

In a recent paper Dias & Vanden-Broeck (1993) presented a model for the spray at the bow of a ship. The bow was assumed to be a semi-infinite flat-bottomed body terminated by a face inclined with the horizontal. The spray was modelled by a layer of water rising along the bow and falling back as a jet (see Figures 1.1a and 1.1b). The solutions were computed by a series truncation procedure. It was found that there is a solution for each value of the Froude number

$$F_d = \frac{U}{(gd)^{1/2}}. \tag{1.1}$$

Here U is the velocity at infinity, g the acceleration due to gravity and d the draft.

For large values of F_d, the separation point D is on the bow as in Figure 1.1a. As F_d decreases, the separation point D rises along the bow and then moves on the free surface. There are therefore two stagnation points S and D on the free surface (see Figure 1.1b). As F_d is further decreased, the distance between the stagnation points D and S increases. However more and more terms in the series representation are needed to compute accurate solutions. Therefore Dias & Vanden-Broeck (1993) presented only solutions for which the distance between the two stagnation points D and S is relatively small.

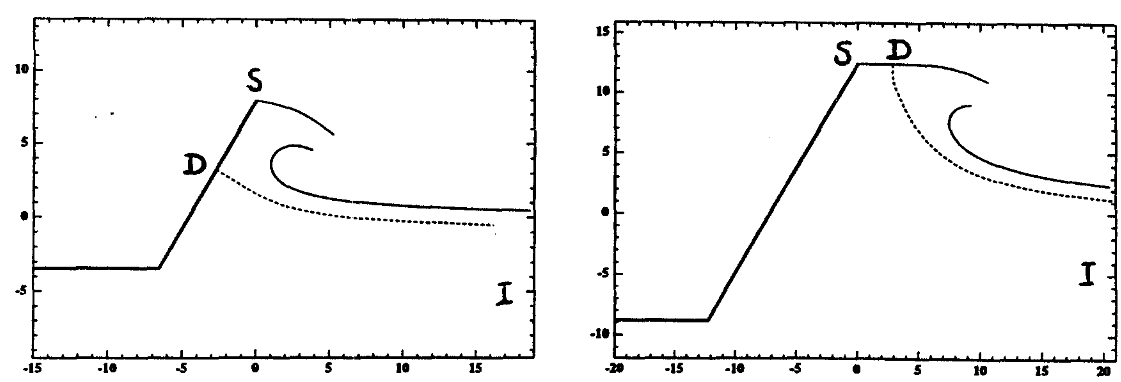
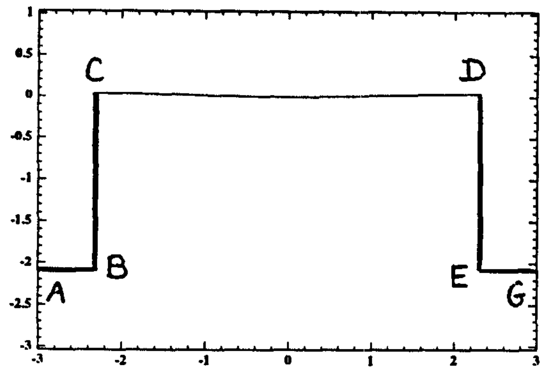


Figure 1.1: Bow flows: (a) stagnation point on the bow, (b) stagnation point on the free surface. The dotted line represents the dividing streamline.



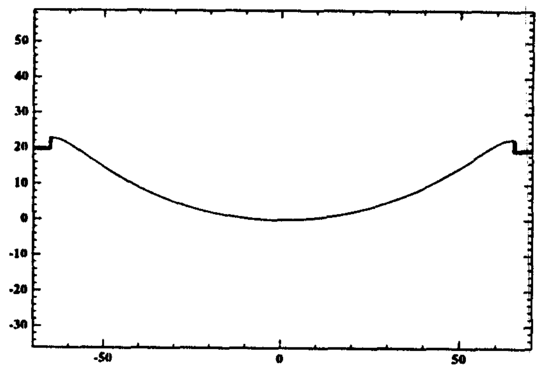


Figure 1.2: Free surface flows with two stagnation points. Computed solutions on branch I with $P = 0.5354$, $\epsilon = 0.5899$ (left) and $P = 75.56$, $\epsilon = 22.14$ (right).

We consider here another flow for which there are two stagnation points on the free surface (see Figure 1.2). This flow can be created by opening a slit in a flow beneath a flat plate and applying a negative pressure in the slit. There is a vertical wall BC and a horizontal wall AB on the left as in Figures 1.1a and 1.1b. There is also a vertical wall ED and a horizontal wall EG on the right. Therefore the configuration of Figure 1.2 can be viewed as an "approximation" of the flow of Figure 1.1b in which the dividing streamline DI is replaced by the walls ED and EG. The "approximation" has the advantage that solutions for which the distance between the two stagnation points is large can be computed.

The computations for large distance between the stagnation points reveal a new feature of the flow. There is a countably infinite number of solutions and the profiles are wavy.

2 Formulation

We consider the flow configuration of Figure 1.2. The flow domain is bounded above by the vertical walls BC and ED and by the horizontal walls AB and EG. We assume that the fluid is incompressible and inviscid and that the flow is steady and irrotational. Far downstream the flow is characterized by a uniform stream with a constant velocity U. We choose cartesian coordinates with the origin on the free surface at an equal distance from the two vertical walls. We assume that the flow is symmetric with respect to the y-axis. We introduce the potential function ϕ and the stream function ψ. Without loss of generality we choose $\psi = 0$ on the free surface and $\phi = 0$ at the origin. We denote by P and $P+K$ the values of the potential function at the points D and E. It follows from the symmetry of the flow that $\phi = -P$ and $\phi = -P-K$ at the points C and B.

We shall construct solutions for which the points C and D are stagnation points. We define dimensionless variables by taking K/U as the unit length and U as the unit velocity. The problem is characterized by the dimensionless value P of the potential at D and the parameter

$$\epsilon = \frac{U^3}{gK}. \tag{2.1}$$

The series expansion formulation consists in mapping the flow domain onto the upper half unit disk and to expand the complex velocity as a Taylor series inside the unit disk. The image of the free surface is the upper half unit circle. The image of the solid boundaries is the real diameter.

The mapping of the flow domain from the plane of the complex potential (f-plane) to the upper half unit disk (t-plane) is provided by

$$f = \frac{e}{(1-e)^2}\left(t + \frac{1}{t}\right), \tag{2.2}$$

with $P = 2e/(1-e)^2$. Next we expand the complex velocity ζ as

$$\zeta = (1-t^2)(e^2-t^2)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} a_n t^{2n}. \tag{2.3}$$

This expansion factors out the singular behavior of the velocity at the corners E and B ($t = \pm e$) and the singular behavior of the velocity at the stagnation points D and C ($t = \pm 1$). Moreover, the expansion takes advantage of the symmetry of the problem. At $t = 0$, the velocity approaches unity. Therefore $a_0 = e$. Bernoulli's equation on the free surface yields

$$\epsilon|\zeta|^2 + 2y = 2y|_{t=\pm 1}. \tag{2.4}$$

Parameterizing the free surface by $t = e^{i\sigma}$, $0 \le \sigma \le \pi$, and differentiating (2.4) with respect to σ leads to

$$\epsilon[u(\sigma)u_\sigma(\sigma) + v(\sigma)v_\sigma(\sigma)] - P \sin\sigma \frac{v(\sigma)}{u^2(\sigma)+v^2(\sigma)} = 0. \tag{2.5}$$

This completes the formulation of the problem. We seek ζ as an analytic function of t in the upper half unit disk, satisfying (2.5).

3 Numerical results

The series is truncated. Let $N-2$ be the order of the truncation, i.e. the coefficients a_0 up to a_{N-2} are kept in the series. First we introduce the mesh points on the free surface

$$\sigma_I = \frac{(I-\frac{1}{2})\pi}{2N-2}, \quad I = 1, N-1. \tag{3.1}$$

We introduce the N unknowns P and $a_I, I = 0, \ldots, N-2$.

We evaluate Bernoulli's equation (2.5) at the mesh points. This provides $(N-1)$ equations. The last equation simply is $a_0 = e$. The system is solved numerically by Newton's method.

We computed eight branches of solutions, which are plotted in Figure 3.1. As P increases along a branch, the waves on the free surface increase in amplitude. One way to characterize each branch is to count the number of waves between the vertical walls. Profiles are shown in Figures 3.2 and 3.3.

As P increases, the flow looks more and more like a stern flow (see for example Figure 1 in Vanden–Broeck (1980)). Figure 3.3 shows a computed solution with $P = 80$ along branch VIII.

REFERENCES

- Dias F. and Vanden–Broeck J.-M. (1993), "Nonlinear bow flows with spray," *J. Fluid Mech.* **255**, pp. 91-102.
- Vanden–Broeck J.-M. (1980), "Nonlinear stern waves," *J. Fluid Mech.* **96**, pp. 603-611.

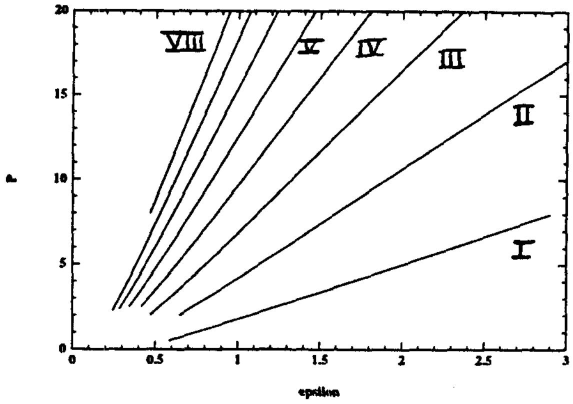


Figure 3.1: Free surface flows with two stagnation points. Computed solutions in the P-ϵ plane. The branches are labelled I, II, etc, counterclockwise. The label is the number of waves of the solution for large P.

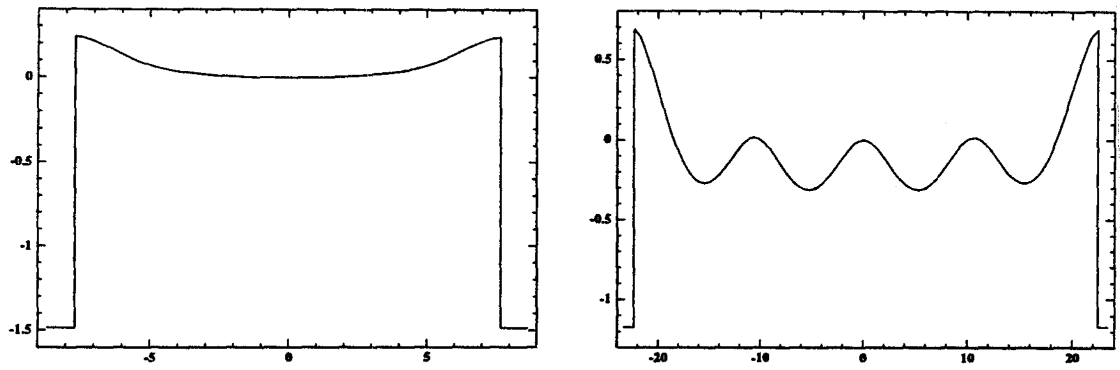


Figure 3.2: Free surface flows with two stagnation points. Computed solutions on branch IV with $P = 5.35\,, \epsilon = 0.64$ (left) and $P = 20\,, \epsilon = 1.80$ (right).

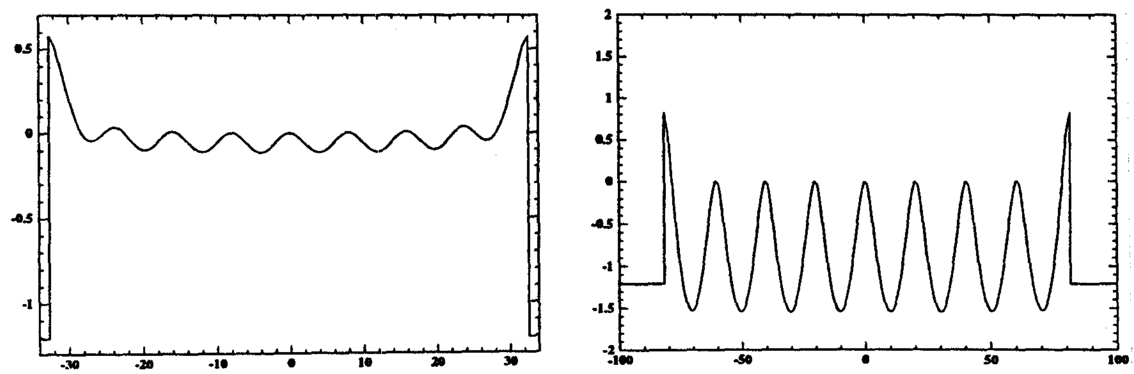


Figure 3.3: Free surface flows with two stagnation points. Computed solution on branch VIII with $P = 30\,, \epsilon = 1.33$ (left) and $P = 80\,, \epsilon = 3.41$ (right).