A VERTICAL MOTION OF A BODY OF REVOLUTION IN A STRATIFIED FLUID

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When moving a body in an inviscid nonhomogeneous incompressible fluid the stratification influence on the resulting hydrodynamic loads comes through variable hydrostatic forces, additional forces caused by energy consumptions for generation of internal gravitational waves and a synchronous response of fluid to the external influence, i.e. the non-gravity fluid reaction.

This paper is concerned with a uniform vertical motion of a slender body of revolution in the infinite fluid with the density distribution as pycnocline. The sharp pycnocline is simulated by the two-layer fluid and the smooth one by the three-layer fluid. The body is supposed to approach the pycnocline from a great distance, and after crossing the pycnocline to recede up to a great distance, the velocity being kept steady throughout. The problem is solved both with regard for the Boussinesq approximation and without it. At the latter case, the load arised from moving the body in the non-gravity fluid is also defined.

The linear problem on hydrodynamic disturbances generating in the stable stratified fluid at the body motion may be solved by the numerical methods based on using a solution for a point source. Let us consider the undisturbed infinite fluid at rest and its density distribution being given by the function $\rho_0(z)$, where the axis z is directed vertically upward. The system of equations describing small motions of this stratified fluid given the mass source of strength $\rho_0 Q(\mathbf{x},t)$ in the system of the Cartesian coordinates $\mathbf{x}=(x,y,z)$ takes the form

$$\rho_0\frac{\partial \mathbf{u}}{\partial t}+\nabla p+\mathbf{F}=0,\quad \frac{\partial \rho}{\partial t}+\rho_0'(z)w=0,\quad div\mathbf{u}=Q(\mathbf{x},t),\quad \mathbf{F}=(0,0,g\rho), \tag{1}$$

$$\mathbf{u},p\to 0\quad (|\mathbf{x}|\to\infty),\quad \mathbf{u}=\rho=0\quad (t=0),$$

where $\mathbf{u}=(u,v,w),p,\rho$ are disturbances of the velocity vector, pressure and density, due to presence of mass-source, g is the gravitational acceleration, and the prime indicates the differentiation with respect to z. For the moving point source $Q(\mathbf{x},t)=q(t)\delta(\mathbf{x}-\mathbf{Y}(t))$ ($q(t)\equiv 0$ if $t\le 0$), δ is the Dirac delta-function, $\mathbf{Y}(t)=(y_1(t),y_2(t),y_3(t))$, $\mathbf{x}=\mathbf{Y}(t)$ is the path of source motion.

The system of equations (1) may be reduced to one equation for the vertical velocity $w(\mathbf{x},t)$

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial z}(\rho_0\frac{\partial w}{\partial z})+\rho_0\Delta_2 w\right]+\rho_0 N^2\Delta_2 w=\frac{\partial^3}{\partial t^2\partial z}(\rho_0 Q), w\to 0(|\mathbf{x}|\to\infty), w=\partial w/\partial t=0(t=0), \tag{2}$$

where $\Delta_2=\partial^2/\partial x^2+\partial^2/\partial y^2$, $N(z)=\sqrt{-g\rho_0'/\rho_0}$ is the buoyancy frequency. All the other variables may be expressed in terms of w by the relations following from (1), in particular, the pressure, calculated without considering hydrostatic forces, is equal to

$$p=\rho_0\Delta_2^{-1}\frac{\partial}{\partial t}(\frac{\partial w}{\partial z}-Q). \tag{3}$$

For solving Eq.(2) the Fourier integral transformation into horizontal variables and a time are used. The equation for the Fourier transform of the vertical velocity w_* is

$$(\rho_0 w_*')'-\rho_0 k^2(1-\frac{N^2}{\omega^2})w_*=(\rho_0 Q_*)',\quad w_*\to 0\quad (|z|\to\infty), \tag{4}$$

of which solution is written by the Green function $G(k,z,\xi,\omega)$ satisfying the equation

$$(\rho_0 G')' - \rho_0 k^2 (1 - \frac{N^2}{\omega^2}) G = \delta(z-\xi), \quad G \to 0 \quad (|z| \to \infty). \tag{5}$$

Following [1], the function G may be represented as the sum of two terms $G(k,z,\xi,\omega) = G_0(k,z,\xi) + G_1(k,z,\xi,\omega)$, where $G_0(k,z,\xi) = \lim_{\omega\to\infty} G(k,z,\xi,\omega)$ is independent of ω and provides a solution of the equation

$$(\rho_0 G_0')' - \rho_0 k^2 G_0 = \delta(z-\xi), \quad G_0 \to 0 \quad (|z| \to \infty). \tag{6}$$

The function G_0 determines an instantaneous part of the fluid response to the external action and describes a part of the disturbance field carrying away by moving source and corresponding to a non-gravity fluid, i.e. a zero vector of the mass forces $\mathbf{F}$ in (1). The remaining part $G_1(k,z,\xi,\omega)$ is a lagging response and describes the internal waves localized nearby the variation density levels. The function G_1 is a solution of the nonuniform equation

$$(\rho_0 G_1')' - \rho_0 k^2 (1 - \frac{N^2}{\omega^2}) G_1 = -\frac{k^2}{\omega^2} \rho_0 N^2 G_0, \quad G_1 \to 0 \quad (|z| \to \infty). \tag{7}$$

The solution of this equation can be represented as a expansion in terms of eigenfunction of the following eigenvalue problem

$$(\rho_0 W_n{}')' - \rho_0 k^2 (1 - \frac{N^2}{\omega^2}) W_n = 0, \quad W_n \to 0 \quad (|z|) \to \infty). \tag{8}$$

The spectrum of this problem $\omega_n^2(k)$ is positive and discrete provided that the function $N(z)$ differs from zero only in finite interval with respect to z. The eigenfunctions $W_n(z)$ are orthogonal and normalized as follows

$$\int_{-\infty}^{\infty} \rho_0(z) N^2(z) W_n^2(z) dz = 1. \tag{9}$$

The system of eigenfunctions is complete in the set of functions vanishing as $|z| \to \infty$. As a result the solution for G_1 is given by

$$G_1 = \sum_{n=1}^{\infty} F_n(\xi) W_n(z), F_n(\xi) = -\frac{\omega_n^2}{\omega_n^2 - (\omega + i\epsilon)^2} \int_{-\infty}^{\infty} \rho_0(\eta) N^2(\eta) G_0(\eta,\xi) W_n(\eta) d\eta, (0 < \epsilon \ll 1).$$

Since the Green function is divided into two parts, the pressure obtained by the inversion Fourier transforms has the form $p = p_0 + p_1$. The term p_0 describes the pressure disturbances in non-gravity fluid and vanishes as the source is shut down

$$p_0(\mathbf{x},t) = \frac{\rho_0(z)}{2\pi} \frac{\partial}{\partial t} \left[q(t) \int_0^{\infty} \frac{dk}{k} J_0(kr(t)) M(z, y_3(t)) \right], \quad M(z,\xi) = \rho_0(\xi) \frac{\partial^2 G_0}{\partial z \partial \xi} + \delta(z-\xi). \tag{10}$$

The term p_1 is the wave part of pressure:

$$p_1(\mathbf{x},t) = -\frac{\rho_0(z)}{2\pi} \int_0^t q(\tau) \rho_0(y_3(\tau)) d\tau \int_0^{\infty} \frac{dk}{k} J_0(kr(\tau)) \times$$

$$\sum_{n=1}^{\infty} \omega_n^2 \cos \omega_n (t-\tau) W_n'(z) \int_{-\infty}^{\infty} \rho_0(\eta) N^2(\eta) \frac{\partial G_0}{\partial \xi}|_{\xi=y_3(\tau)} W_n(\eta) d\eta, \tag{11}$$

where $r(\zeta) = [(x - y_1(\zeta))^2 + (y - y_2(\zeta))^2]^{1/2}, J_0$ is the first kind of the Bessel function of zero order.

In a weakly stratified fluid we can introduce the Boussinesq approximation, where the variation of the density from some constant value, for example $\rho_s = \rho_0(0)$, takes into account only for the buoyancy term in the equation of the momentum conservation (1). In the inertia term the real density is substituted by the value ρ_s . In this approximation Eq.(2) takes the form:

$$\frac{\partial^2}{\partial t^2}\Delta w + N^2\Delta_2 w = \frac{\partial^3 Q}{\partial t^2 \partial z},$$

with the buoyancy frequency is now equal to $N(z) = \sqrt{-g\rho_0'/\rho_s}$ and Δ is the 3-D Laplace operator. In the relationships (3-9) $\rho_0(z)$ should be replaced by ρ_s. In this case Eq.(6) has a simple solution corresponding to the homogeneous fluid $G_0 = -\exp(-k|z-\xi|)/(2\rho_s k)$. On integrating (10) the well-known result for the infinite homogeneous fluid is obtained $p_0(\mathbf{x}, t) = (\rho_s/4\pi)\partial(q(t)/r_1)/\partial t$, where $r_1 = [r^2(t) + (z - y_3(t))^2]^{1/2}$.

Furthermore, the uniform vertical motion of the constant strength source at the velocity U is only considered. It is convenient to make a time shift. The instant at which the source coincides with the origin of coordinates is taken as an initial time. Then the source path has the form $y_1(t) = y_2(t) = 0, y_3 = Ut, r^2 = x^2 + y^2$ and the flow becomes axially symmetric.

As known the axially directed motion of the slender body of revolution into the infinite homogeneous fluid can be simulated by the motion of singularities located continuously on the body axis. Let in the moving coordinate system $x_1 = x, y_1 = y, z_1 = z - Ut$ the body surface be given by $r = f(z_1)$. Denote a half-width and half-length of the body by a and b respectively, for the slender body $a/b \ll 1$. The singularity system equivalent to this body has the following distribution over the interval $|z_1| \le b$: $q(z_1, t) = -2\pi U f(z_1) f'(z_1) = -US'(z_1)$, where $S = \pi f^2$ is a cross section area.

The total pressure in fluid due to the motion of this singularity system is equal to

$$P(r, z, t) = \int_{-b}^{b} q(s, t) p(r, z, t, s) ds,$$

with substituting $q(\zeta) = 1, y_1 = y_2 = 0, y_3(\zeta) = s + U\zeta$ in Eqs.(10),(11) for p_0 and p_1.

The vertical force R acting on the slender body is found by integrating the pressure over the body surface. Without introducing the Boussinesq approximation the vertical force is defined as the sum of two terms $R = R_0 + R_1$, the former corresponds to the motion of body in a non-gravity fluid and the latter describes the wave effect. Under the Boussinesq approximation the vertical force is determined only by the wave component of the point source pressure because of the resistance of the translating body is equal to zero in the infinite homogeneous fluid (the d'Alembert paradox).

The simplest example of a stratified fluid is the two-layer fluid. The upper layer of the density ρ_1 occupies the region $z > 0$ and the lower one of the density $\rho_2 = (1+\varepsilon)\rho_1$ $(\varepsilon > 0)$ occupies the region $z < 0$. The particular case of this fluid with $\varepsilon \to \infty$ is the infinite homogeneous fluid with the free surface. By [1] the solution of Eq.(6) is $G_0(k, z, \xi) = -\left[e^{-k|z-\xi|} - \gamma sgn\xi e^{-k(|z|+|\xi|)}\right]/(2k\rho_0(\xi))$, with $\gamma = \varepsilon/(2+\varepsilon)$. The eigenvalue problem (8) has a unique solution and there is a wave mode alone in this fluid

$$W_1 = e^{-k|z|}/\sqrt{2\bar{g}\rho_s}, \quad \omega_1 = \sqrt{\bar{g}k}, \quad \bar{g} = \gamma g, \quad \rho_s = \rho_1(2+\varepsilon).$$

The forces acting on the slender body moving into the two-layer fluid are

$$R_0 = -\frac{U^2}{4\pi}\left\{\int_{-b}^{b} \rho_0(\eta + Ut)S'(\eta)d\eta \int_{-b}^{b} \frac{(\eta-\zeta)S'(\zeta)d\zeta}{[f^2(\eta) + (\eta-\zeta)^2]^{3/2}} - \gamma \int_{-b}^{b} \rho(\eta+Ut)sgn(\eta+Ut)\times\right.$$

$$S'(\eta)d\eta\int_{-b}^{b}\frac{(|\eta+Ut|+|\zeta+Ut|)}{[f^2(\eta)+(|\eta+Ut|+|\zeta+Ut|)^2]^{3/2}}sgn(\zeta+Ut)S'(\zeta)d\zeta\Big\},$$

$$R_1=-\frac{\bar{g}U}{4\pi}\int_{-b}^{b}\rho_0(Ut+\eta)sgn(\eta+Ut)S'(\eta)d\eta\int_{-b}^{b}S'(\zeta)d\zeta\times$$

$$\int_{-\infty}^{t}[sgn(\zeta+U\tau)-\gamma]d\tau\int_0^\infty kJ_0(kf(\eta))e^{-k(|\eta+Ut|+|\zeta+U\tau|)}\cos\omega_1(t-\tau)dk. \quad (12)$$

In the three-layer fluid the undisturbed continuous distribution of density has the exponentially stratified intermediate layer $|z|<H$ with $\rho_0(z)=\rho_s e^{-\alpha z}$ and the homogeneous upper and lower layers.The buoyancy frequency differs from zero only in the intermediate layer wherein it is the constant value $N=\sqrt{\alpha g}$. In this case Eq.(6) has constant coefficients and an analytic solution. Eigenvalue problem (8) has been studied in detail in [2].

Without introducing the Boussinesq approximation the final solution for the three-layer fluid is

$$R_0=-\frac{U^2}{2\pi}\int_{-b}^{b}\rho_0(\eta+Ut)S'(\eta)d\eta\int_{-b}^{b}S'(\zeta)d\zeta\int_0^\infty\frac{dk}{k}J_0(kf(\eta))\times$$

$$M_1(\eta+Ut,\zeta+Ut)dk,\quad M_1(z,\xi)=\partial M(z,\xi/\partial\xi,$$

$$R_1=\frac{UN^2}{2\pi}\sum_{n=1}^{\infty}\int_{-b}^{b}\rho_0(\eta+Ut)S'(\eta)d\eta\int_{-b}^{b}S'(\zeta)d\zeta\int_{-\infty}^{t}\rho_0(\zeta+U\tau)d\tau\times$$

$$\int_0^\infty\frac{dk}{k}J_0(kf(\eta))\omega_n^2\Phi_n(\zeta+U\tau)W_n'(\eta+Ut)\cos\omega_n(t-\tau),\quad \Phi_n(\xi)=\int_{-H}^{H}\rho_0(s)\frac{\partial G_0(s,\xi)}{\partial\xi}W_n(s)ds. \quad (13)$$

Under the Boussinesq approximation expression (13) for the wave component of vertical force is rather simple:

$$R_1=-\frac{\rho_s^2UN^2}{4\pi}\sum_{n=1}^{\infty}\int_{-b}^{b}S'(\eta)d\eta\int_{-b}^{b}S'(\zeta)d\zeta\int_{-\infty}^{t}d\tau\int_0^\infty\frac{J_0(kf(\eta))}{k}\times$$

$$\omega_n^2\Psi_n(\zeta+U\tau)W_n'(\eta+Ut)\cos\omega_n(t-\tau)dk,\quad \Psi_n(\xi)=\int_{-H}^{H}e^{-k|s-\xi|}sgn(s-\xi)W_n(s)ds. \quad (14)$$

For the slender body in Eqs.(13),(14) the Bessel function may be approximately taken as 1 .

Three bodies with different shapes are taken for numerical computations:

$$f(z_1)=a\sqrt{1-z_1^2/b^2}\quad(B1), f(z_1)=a\cos(\pi z_1/2b)\quad(B2), f(z_1)=a(1+\cos(\pi z_1/b))/2\quad(B3). \quad (15)$$

The spheroid B1 is the blunt body. The body B2 has an acute angle at the top and the body B3 has a zero angle.

The body shape selection in form (15) along with simple models of density distribution allows some of the integrations in (12)-(14) to be carried out analytically. The expressions obtained are reduced into double integrals. The components R_0 and R_1 of the vertical force in the two-layer and three-layer fluid are determined. The influence of body shape, body velocity, pycnocline depth, and density difference in the pycnocline on the lift force are studied.

References

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