

INTERACTION BETWEEN SOLITARY WAVE AND A FLOATING ELASTIC PLATE — 3-D Theory —

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INTRODUCTION

The author (1996) derived the Boussinesq class equations which represent a long wave motion beneath a two-dimensional floating elastic plate, and discussed matching conditions between the ordinary Boussinesq's equations and the above mentioned modified Boussinesq's equations by employing the matched asymptotic expansion method in the case of the two-dimensional model.

In the present paper, this theory is extended to the three-dimensional theory. The three-dimensional Boussinesq class equations and the three-dimensional matching conditions are discussed.

THEORY

The overall problem is divided into three parts. The first one is an ordinary shallow water wave problem which is defined in the region I . The second one is a problem of shallow water waves in a particular region which is covered with an elastic plate. The third one is defined at the intersection between the free surface and the edge of the elastic plate which is the singular point of the above mentioned two regions. The first two are called outer problems and the third is an inner problem which is defined at a singular point of outer problems. The overall solution is obtained by a matching procedure.

Outer problem

In order to simplify these problems, it is assumed that the flow is incompressible and irrotational. Therefore the velocity potential is introduced. For describing irrotational flow, the following nondimensional variables are introduced.

$$(x, y, z) = \left(\frac{X}{\lambda}, \frac{Y}{h}, \frac{Z}{h} \right), \quad t = \frac{\hat{c}T}{\lambda}, \quad \phi = \frac{\Phi}{\hat{c}\lambda} \quad (1)$$

$$p = \frac{P - P_a}{\rho gh}, \quad \bar{D} = \frac{D}{\rho g \lambda^4}, \quad w = \frac{W}{\rho gh}, \quad \zeta = \frac{\hat{\zeta}}{h} \quad (2)$$

Where ; λ = a characteristic wave length ; h = the water depth ; $\hat{c} = \sqrt{gh}$ a typical wave speed ; Φ = the velocity potential ; P = the water pressure ; P_a = the ambient surface pressure ; ρ = the density of water ; D = the flexural rigidity of the elastic plate ; W = the weight per unit area of the elastic plate ; and $\hat{\zeta}$ = the free surface displacement. Usually, the shallow water problem is characterized by two important parameters:

$$\beta \equiv \left(\frac{h}{\lambda} \right)^2, \quad \alpha \equiv \frac{a}{h} = O(\beta) \quad (3)$$

where a = a representative wave amplitude and, by definition, $\beta \ll 1$. The order assumption of equation (3) leads to the Boussinesq class equations for shallow water waves (Wu 1981). The unit weight of the elastic plate is assumed to be such that the draft of the elastic plate is the same order of the wave amplitude and an order for the nondimensional flexural rigidity is assumed as the following equation.

$$w = O(\alpha), \quad \bar{D} = o(1) \quad (4)$$

First of all, let us consider the pressure induced by the deformation of the elastic plate floating on the water of the region II . Since the previous order assumption implies that the influence of mass of the elastic plate is negligible, the pressure is given as follows.

$$p = w + \bar{D} (\nabla^2 \cdot \nabla^2) \zeta + o(\alpha^2, \alpha\beta, \beta^2) \quad (5)$$

Substituting equation (5) into the original expression of the Boussinesq's equation, the following equations are obtained.

$$\zeta_t + \nabla [(1 + \zeta) \nabla \phi] = 0 \quad (6)$$

$$\phi_t + \frac{1}{2} (\nabla \phi)^2 + \zeta + w + \bar{D} (\nabla^2 \cdot \nabla^2) \zeta - \frac{\beta}{3} \nabla^2 \phi_t = 0 \quad (7)$$

Equations (6) and (7) are the Boussinesq class equations for shallow water waves covered with an elastic plate. The Boussinesq class equations for the region I which is not covered with the elastic plate are easily obtained by eliminating the fourth and the fifth term of equation (7).

Inner Problem

It is obvious that the inner region has to include the sea floor boundary to get the Boussinesq class solution. Therefore, a local coordinate system is introduced and its nondimensionalization should be:

$$(n, s) = \frac{(N, S)}{\lambda}, \quad \hat{n} = \frac{N}{h}. \quad (8)$$

The nondimensionalization of the velocity potential in the inner region and the radial of curvature are represented as follows:

$$\phi_I = \frac{\Phi}{h\hat{c}}, \quad \kappa = \frac{K}{\lambda}. \quad (9)$$

In order to obtain the inner solution, the following asymptotic expansion of the velocity potential and the free surface elevation are employed.

$$\phi_I = -\beta^{-\frac{1}{2}} \alpha \int_0^t C(s, t) dt + \alpha \phi_{I0} + \beta^{\frac{1}{2}} \alpha \phi_{I1} + \dots \quad (10)$$

$$\zeta_I = \alpha \zeta_{I0} + \beta^{\frac{1}{2}} \alpha \zeta_{I1} + \alpha \beta \zeta_{I2} + \dots \quad (11)$$

Substituting (10) into the Laplace's equation and eliminating higher order terms, the governing equations of the first and the second order velocity potential are obtained respectively.

$$\phi_{I0\hat{n}\hat{n}} + \phi_{I0z z} = 0 \quad (12)$$

$$\phi_{I1\hat{n}\hat{n}} + \phi_{I1z z} - \int_0^t C_{ss} dt + \frac{1}{\kappa} \phi_{I0\hat{n}} = 0 \quad (13)$$

Substituting equations (10) and (11) into the free surface conditions and eliminating higher order terms, the following boundary conditions of the first and the second order problems are obtained.

$$\phi_{I0z} = 0 \quad \text{on } z = \alpha C \quad (14)$$

$$\phi_{I0z} = 0 \quad \text{on } z = -1 \quad (15)$$

$$\phi_{I1z} = C_t \quad \text{on } z = 0 \quad (16)$$

$$\phi_{I1z} = 0 \quad \text{on } z = -1 \quad (17)$$

The asymptotic expansion of the inner variable of the pressure term Λ_I can be assumed as follows:

$$\Lambda_I = \alpha \Lambda_{I0} + \beta^{\frac{1}{2}} \alpha \Lambda_{I1} + \dots \quad (18)$$

where the outer variable of the pressure term is represented as follows:

$$\Lambda = \zeta + w + \bar{D} (\nabla^2 \cdot \nabla^2) \zeta. \quad (19)$$

Substituting the asymptotic expansion of Λ_I into the dynamic free surface condition and after some manipulations, boundary conditions for the first and the second order problem are obtained as follows:

$$\phi_{I0z} = 0 \quad \text{on } z = \alpha C - w \quad (20)$$

$$\phi_{I0z} = 0 \quad \text{on } z = -1 \quad (21)$$

$$\phi_{I1z} = C_t \quad \text{on } z = 0 \quad (22)$$

$$\phi_{I1z} = 0 \quad \text{on } z = -1. \quad (23)$$

The first order problem is easily solved by means of the Schwarz-Christoffel transformation, and its eigen function expansion form is represented as follows:

$$\phi_{I0} = u_0 \hat{n} + \sum_{m=1}^{\infty} a_m \cos m\pi \frac{z+1}{1+\alpha C} \exp\left(-\frac{m\pi}{1+\alpha C} \hat{n}\right) \quad \text{in Region I} \quad (24)$$

$$\phi_{I0} = u_1 \hat{n} + \sum_{m=1}^{\infty} a_m \cos m\pi \frac{z+1}{1+\alpha C - w} (z+1) \exp\left(-\frac{m\pi}{1+\alpha C - w} \hat{n}\right) \quad \text{in Region II.} \quad (25)$$

The relationship between the normal velocity of region I u_0 and that of region II u_1 is presented as follows:

$$u_1 = \gamma u_0, \quad \gamma = \frac{1 + \zeta(0_+, s, t)}{1 + \zeta(0_-, s, t)}. \quad (26)$$

It is difficult to obtain the complete solution for the second order problem, but the leading term of the second order solution is easily obtained as follows:

$$\phi_{I1} = C_t \left(\frac{1}{2} z^2 + z\right) - \frac{1}{2} \left(C_t - \int_0^t C_{ss} dt + \frac{u_0}{\kappa}\right) \hat{n}^2 + O(\alpha). \quad (27)$$

Matching

Region I : The matching region should be a region of $n = O(\beta^{\frac{1}{2}-\delta})$ where $0 < \delta < \frac{1}{6}$. Therefore the depth mean outer expansion of the inner solution of the velocity potential is represented as follows:

$$\bar{\phi}_I = -\alpha\beta^{-\frac{1}{2}} \int_0^t C dt + \alpha u_0 \hat{n} - \frac{\alpha\beta^{\frac{1}{2}}}{2} \left(C_t - \int_0^t C_{ss} dt + \frac{u_0}{\kappa}\right) \hat{n}^2 - \frac{\alpha\beta^{\frac{1}{2}}}{3} C_t + o(\alpha\beta^{\frac{1}{2}}, \alpha^{\frac{1}{2}}\beta, \alpha^{\frac{3}{2}}, \beta^{\frac{3}{2}}). \quad (28)$$

Similarly, the outer expansion of the inner solution of the free surface elevation is presented as follows:

$$\zeta_I = \alpha C - \alpha\beta^{\frac{1}{2}} u_0 \hat{n} - \frac{\alpha\beta}{2} \left(C_{tt} - C_{ss} + \frac{u_0 t}{\kappa}\right) \hat{n}^2 - \frac{\alpha^2}{2} \left\{u_0^2 + \left(\int_0^t C_s dt\right)^2\right\} + o(\alpha^2, \alpha\beta, \beta^2). \quad (29)$$

Since the inner expansion of outer solution is presented by Taylor's expansion, matching conditions are presented as follows:

$$\phi = -\alpha \int_0^t C dt - \frac{\alpha\beta}{3} C_t \quad (30)$$

$$\phi_n = \alpha u_0 \quad (31)$$

$$\phi_{nn} = -\alpha C_t + \alpha \int_0^t C_{ss} dt - \alpha \frac{u_0}{\kappa} \quad (32)$$

$$\zeta = \alpha C - \frac{\alpha^2}{2} \left\{u_0^2 + \left(\int_0^t C_s dt\right)^2\right\}. \quad (33)$$

Region II : In this region, the outer expansion of the inner solution of the velocity potential is almost the same as that of region I as shown below.

$$\bar{\phi}_I = -\alpha\beta^{-\frac{1}{2}} \int_0^t C dt + \alpha u_1 \hat{n} - \frac{\alpha\beta^{\frac{1}{2}}}{2} \left(C_t - \int_0^t C_{ss} dt + \frac{u_0}{\kappa} \right) \hat{n}^2 - \frac{\alpha\beta^{\frac{1}{2}}}{3} C_t + o(\alpha\beta^{\frac{1}{2}}, \alpha^{\frac{1}{2}}\beta, \alpha^{\frac{3}{2}}, \beta^{\frac{3}{2}}) \quad (34)$$

The outer expansion of Λ_I is also obtained as follows by a similar method.

$$\Lambda_I = \alpha C - \alpha\beta^{\frac{1}{2}} u_1 \hat{n} - \frac{\alpha\beta}{2} \left(C_{tt} - C_{ss} + \frac{u_0 t}{\kappa} \right) \hat{n}^2 - \frac{\alpha^2}{2} u_1^2 + o(\alpha^2, \alpha\beta, \beta^2) \quad (35)$$

The outer expansion of the inner solution of the free surface elevation can not be obtained exactly up to the second order. But, from the assumption of (4), the leading term of the free surface elevation is as follows:

$$\zeta_I = \alpha C - w + o(\alpha, \beta) \quad (36)$$

After the matching between the outer solutions and the inner solutions, and taking the end condition at the tip of the elastic plate into account, the following equations are obtained.

$$\phi = -\alpha \int_0^t C dt - \frac{\alpha\beta}{3} C_t \quad (37)$$

$$\phi_n = \alpha u_1 \quad (38)$$

$$\phi_{nn} = -\alpha C_t + \alpha \int_0^t C_{ss} dt - \alpha \frac{u_0}{\kappa} \quad (39)$$

$$\zeta + \bar{D} (\nabla^2 \cdot \nabla^2) \zeta = \alpha C - w - \frac{\alpha^2}{2} \left\{ u_1^2 + \left(\int_0^t C_s dt \right)^2 \right\} \quad (40)$$

$$(\nabla^2 \zeta)_n + (1 - \nu) \zeta_{ssn} = 0 \quad (41)$$

$$\zeta_{nn} + \nu \zeta_{ss} = 0 \quad (42)$$

NUMERICAL SCHEME

The numerical scheme which is used in this paper is the finite difference method. The Crank-Nicolson method is used for the time marching scheme and the SOR is applied to solve the system of linear equations. The main feature of the present scheme is that the mass conservation form is introduced into the first equation of the Boussinesq's equations and the truncation errors of those numerical schemes are $O(\Delta t^3)$ and $O(\Delta x^3)$.

CONCLUSIONS

The three-dimensional modified Boussinesq's equations which represent a long wave beneath an elastic plate have been derived. Applying the matched asymptotic expansion method, conditions for a connection between the solution of the ordinary Boussinesq's equations and the modified Boussinesq's equations have also been obtained. These sets of equations were then solved by the finite difference method, and numerical results of the behavior of a solitary wave beneath an elastic plate have been obtained.

REFERENCE

- Takagi, K. (1996). "Interaction between Solitary Wave and Floating Elastic Plate." *J. Wtrwy., Port, Coast., and Oc. Engrg., ASCE*, (to appear)
- Wu, T. Y. (1981) "Long Waves on Ocean and Coastal Waters." *J. Engrg. Mech., ASCE*, 107(3), 501-522.

DISCUSSION

Schultz: You present an example where the soliton runs along the side of the elastic plate. Have you performed the normally incident wave, to find out how the soliton enters beneath the elastic plate?

Takagi: It is found that the reflection of the soliton along the side of the elastic plate is small. Some examples of the 2-D numerical simulation can be found in:

Takagi, K., *Interaction between solitary wave and floating elastic plate*, J. Wtrwy, Port, Coast., and Oc. Engrg., ASCE (to appear)