An effective method for second order potential

Towards the calculation of third order forces

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1. Introduction

It is wounded that the third order force is an exciting source for the "ringing" phenomenon, observed on some offshore structures. The calculation of third order force needs the information of first and second order potentials on the whole free surface. For the second order problem, Kim and Yue (1990) and Eatock Taylor and Chau (1992) used direct methods to calculate forces after resolving the second order potential on body surface. But for the second order potential on the whole free surface, the method is very expensive and not practical.

Another method proposed by Kriebel (1990) is to represent the second order potential in an explicit form. After getting a wave number spectrum, the second order potential at any point can be determined easily. Then, third order forces can be obtained by an indirect method which is an analogue to the indirect method for second order force. In this paper, further analysis on the wave number spectral function is made. It is found that there are three waves with different wave lengths in its integrand, and the function goes to infinity at the wave number of twice of the incident wave number when water depth is not infinity. Based on the analysis, effective methods for calculating the spectral function and removing the singularity are proposed.

2. Formulation

We divide the second order potential into the incident potential ϕ^I, the particular solution ϕ^p, which satisfies the inhomogeneous boundary condition

$$g\phi_z^{(2)p} - 4\omega^2\phi^{(2)p} = f^{IS}(kr,\theta) + f^{SS}(kr,\theta) = -\frac{ig^2k^2A^2}{2\omega}\sum_{m=0}^{\infty}[f_m^{IS}(kr) + f_m^{SS}(kr)]\cos m\theta \tag{1}$$

on the free surface, where ω is wave frequency and A is the wave amplitude, and the homogeneous potential ϕ^h, which must be determined by the boundary condition on the body surface.

The inhomogeneous potential is obtained by integrating the forcing on the free surface with Green's function

$$\begin{aligned}\phi^{(2)p} = -i\frac{gkA^2}{2\omega}\sum_{m=0}^{\infty}\cos m\theta\Big[i2\pi\frac{\cosh k_2 d\, D_m(k_2)}{\sinh 2k_2 d + 2k_2 d}\cosh k_2(d+z)J_m(k_2 r) \\ + PV\int_0^{\infty}\frac{D_m(\mu)}{\cosh\mu d(\mu\tanh\mu d - k_2\tanh k_2 d)}\cosh\mu(d+z)J_m(\mu r)\,d\mu\end{aligned} \tag{2}$$

where

$$D_m(\mu) = \mu k\int_a^{\infty} r_0[f_m^{IS}(kr_0) + f_m^{SS}(kr_0)]J_m(\mu r_0)\,dr_0 \tag{3}$$

k, k_2 are the real roots of wave dispersion equation at wave frequency and double wave frequency, and a is the radius of water plan of an axisymmetric body. The homogeneous solution can be easily obtained by assuming the particular potential as an incident potential.

3. Analysis on D_m

The calculation of D_m is the key to the present method. By some transform, the spectral function D_m can be represented as

$$D_m(\mu) = \mu k\int_a^{\infty} f_m(kr)J_m(\mu r)\,r\,dr = D_{am}(\mu) + D_{\infty m}(\mu) \tag{4}$$

where

$$D_{a0}(\mu)=\frac{\mu}{k}\sum_{n=1}^{\infty}[S_1(n,n,a)J_0(\mu a)+S_2(n,n,a)J_0'(\mu a)\mu a]+2[S_1(0,0,a)J_0(\mu a)+S_2(0,0,a)J_0'(\mu a)\mu a]$$

$$D_{am}(\mu)=2\frac{\mu}{k}\sum_{n=m}^{\infty}[S_1(n,n-m,a)J_m(\mu a)+S_2(n,n-m,a)J_m'(\mu a)\mu a] \tag{5}$$

$$+\sum_{n=0}^{m}[S_1(n,m-n,a)J_m(\mu a)+S_2(n,m-n,a)J_m'(\mu a)\mu a] \qquad m>0$$

$$D_{\infty 0}=(k^2+3k^2\tanh^2 kd-\mu^2)\frac{\mu}{k}\int_a^{\infty}[\sum_{n=1}^{\infty}S_2(n,n,r)+2S_2(0,0,r)]J_0(\mu r)rdr$$

$$D_{\infty m}=(k^2+3k^2\tanh^2 kd-\mu^2)\frac{\mu}{k}\int_a^{\infty}[2\sum_{n=m}^{\infty}S_2(n,n-m,r)+\sum_{n=0}^{m}S_2(n,m-n,r)]J_m(\mu r)rdr \qquad m>0 \tag{6}$$

$$S_1(n,m,r)=\frac{1}{2}\varepsilon_n\varepsilon_m i^n i^m[\alpha_n J_m'(kr)H_n(kr)+\alpha_n J_m(kr)H_n'(kr)+\alpha_m J_n'(kr)H_m(kr)+$$

$$\alpha_m J_n(kr)H_m'(kr)-\alpha_n\alpha_m H_n'(kr)H_m(kr)-\alpha_n\alpha_m H_n(kr)H_m'(kr)]kr \tag{7}$$

$$S_2(n,m,r)=\frac{1}{2}\varepsilon_n\varepsilon_m i^n i^m[\alpha_n\alpha_m H_n(kr)H_m(kr)-\alpha_n J_m(kr)H_n(kr)-\alpha_m J_n(kr)H_m(kr)]$$

When kr is big, Hankel function can be approximated by its asymptotic representation

$$H_n(kr)=\sqrt{\frac{2}{\pi kr}}\,e^{i(kr-\gamma_n)}\sum_{l=0}^{M_n}C_{nl}(kr)^{-l} \qquad \gamma_n=\frac{n\pi}{2}+\frac{\pi}{4} \tag{8}$$

The integral of triple product of Hankel and Bessel functions from a big value b are

$$I_{lmn}^{l}(b,\infty)=\frac{\mu}{k}\int_b^{\infty}rJ_l^l(kr)H_m^l(kr)J_n^l(\mu r)dr=\frac{\sqrt{\mu}}{4k^2}\left(\frac{2}{\pi}\right)^{3/2}C_{l0}C_{m0}C_{n0}\int_b^{\infty}[e^{i((2k+\mu)r-\gamma_l-\gamma_m-\gamma_n)}$$

$$+e^{i((2k-\mu)r-\gamma_l-\gamma_m-\gamma_n)}+e^{i(\mu r+\gamma_l-\gamma_m-\gamma_n)}+e^{-i(\mu r-\gamma_l+\gamma_m-\gamma_n)}]\sum_{p=0}^{\infty}(kr)^{-p}\sum_{q=0}^{\infty}(kr)^{-q}\sum_{s=0}^{\infty}(kr)^{-s}\frac{dr}{\sqrt{r}} \tag{9}$$

It can be seen that in the integrand there are three waves with wave numbers of $2k+\mu$, $2k-\mu$ and μ. The leading term of the above integral has the form

$$I_{lmn}^{l}(b,\infty)=\frac{\mu}{k}\int_a^{\infty}rJ_l^l(kr)H_m^l(kr)J_n^l(\mu r)dr=\frac{1}{k^2\pi}C_{l0}C_{m0}C_{n0}\Big[\sqrt{\frac{\mu}{2k+\mu}}\Big(\frac{(1+i)}{2}-C_2((2k+\mu)b)-$$

$$iS_2((2k+\mu)b)\Big)e^{i(-\gamma_l-\gamma_m-\gamma_n)}+\sqrt{\frac{\mu}{|2k-\mu|}}\Big(\frac{1\pm i}{2}-C_2(|2k-\mu|b)\mp iS_2(|2k-\mu|b)\Big)e^{i(-\gamma_l-\gamma_m+\gamma_n)}+ \tag{10}$$

$$\Big(\frac{1+i}{2}-C_2(\mu b)-iS_2(\mu b)\Big)e^{i(\gamma_l-\gamma_m-\gamma_n)}+\Big(\frac{1-i}{2}-C_2(\mu b)+iS_2(\mu b)\Big)e^{-i(-\gamma_l+\gamma_m-\gamma_n)}\Big]$$

The signs in the second term of the right-hand side depend on if μ is larger than $2k$ or not. It can be seen that I_{lmn} approaches a constant when μ approaches 0, and to infinity with a speed of $1/|2k-\mu|^{1/2}$ when u approaches 2k. However, for the spectral function D_m the singularity can be canceled by the coefficient in equation (6) when water depth is deep. In finite water depth, the singularity at $2k$ can be removed by changing variables $t^2=|2k-\mu|$. For the convenience in numerical calculation, the following method is applied

$$\int_{2k-\sqrt{k}}^{2k+\sqrt{k}}F(\mu)D(\mu)d\mu=\int_{2k-\sqrt{k}}^{2k+\sqrt{k}}F(\mu)[D(\mu)-D^s(\mu)]d\mu+\int_0^{\sqrt{k}}[F(2k+t^2)D^s(2k+t^2)+F(2k-t^2)D^s(2k-t^2)]2tdt \tag{11}$$

for the integration at the vicinity of $2k$.

Numerical Examination

The integrand of the spectral function D_m is an oscillating function with the increase of the distance from the body, and its amplitude decays slowly at the rate of $1/r^{1/2}$. The earliest computing method is just to truncate at a large radius. Later, a technique was proposed to separate integrating domain into near and far field domains, and in the far field to integrate to infinity analytically after using the asymptotic representation of Hankel function. However, for the present problem, when μ is very small or μ approaches $2k$, the longest wave length is very long, and for a convergent result the radius of the near field should be so big that the method is hardly to be used.

Figures 1 to 4 show the mode 1 and 5 of D_m at μ/k=0.2, gotten by different methods. It can be seen that to truncate the integration domain at will will introduce a big error even when R/a is very big, say $R/a=160$. However, with the increase of the radius of integrating domain, the integral oscillates about and converges to the value obtained by integrating to infinity. The method used in this research is to take an average of the function in one wave length of the longest wave. It can be seen from the figures that when the starting point is not small, the averaged value is very close to the value obtained by integrating to infinity. This method can also be used for the second order difference frequency problem in bichromatic waves, when difference frequency is very small and the asymptotic representation of Hankel function is hardly used.

Figures 5 and 6 show the distribution of the mode 1 of D_m versus wave number. It can be seen that when truncating at $R=30a$ there are a lot of oscillation, but are very smooth after averaging. It can also be seen that at $\mu=0$ the function goes to a constant and to infinity at $\mu=2k$. The imaginary part of D_m also goes to infinity in different direction when μ approaches $2k$ from different side.

After getting the wave number spectrum, second order potentials and forces can be obtained easily. The Table 1 shows the comparison with the Eatock Taylor and Hung's (1987) on the part of second order force from the particular solution and its corresponding homogenous potential. For the cylinder with a draft of h/a=1, a mesh of 16 (4x4) elements on a quadrant of the cylinder is applied, and a mesh of 24 (4x6) elements is applied for the cylinder with a draft of h/a=10. It can be seen the results agree each other.

Table 1. Comparison on second order forces

h/a	*ka*	Present Results	Eatock Taylor's[1]
1.0	1.0	.8930E04 .8553E04	.1016E05 .7490E04
	2.0	.2336E05 -.1485E05	.2311E05 -.1614E05
10.0	1.0	.1849E05 .2744E05	.2099E05 .2670E05
	2.0	.4634E05 -.2612E05	.4829E05 -.2977E05

References

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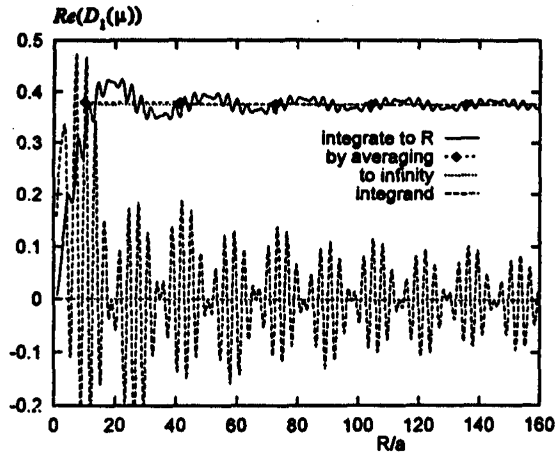

Fig. 1 The examination of the spectral function D_1 for a uniform cylinder obtained by different methods, at a=0.2, ka=1.0 and h/a=1.0.

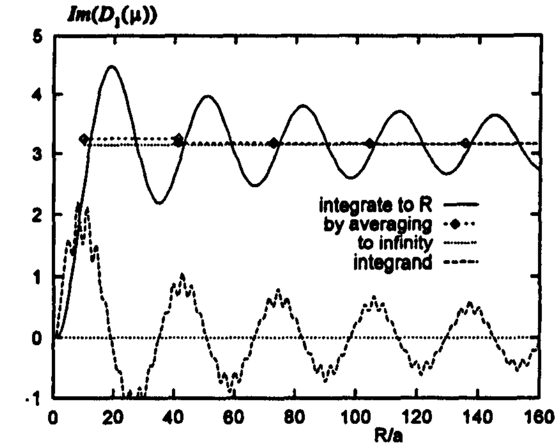

Fig. 2 The examination of the spectral function D_1 for a uniform cylinder obtained by different methods, at a=0.2, ka=1.0 and h/a=1.0.

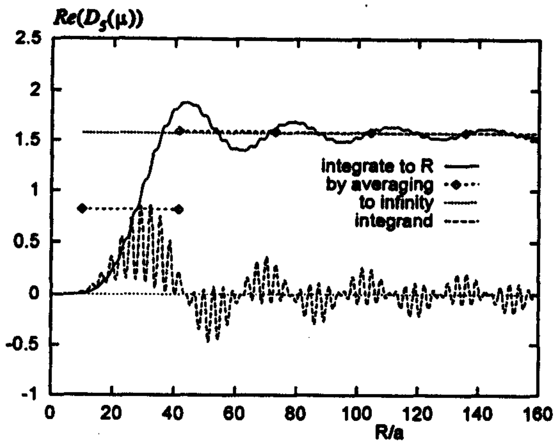

Fig. 3 The examination of the spectral function D_5 for a uniform cylinder obtained by different methods, at a=0.2, ka=1.0 and h/a=1.0.

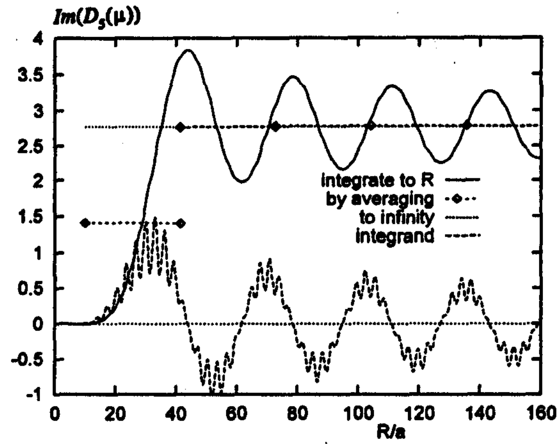

Fig. 4 The examination of the spectral function D_5 for a uniform cylinder obtained by different methods, at a=0.2, ka=1.0 and h/a=1.0.

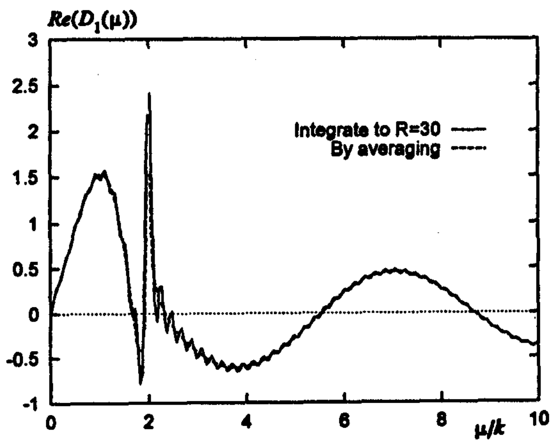

Fig. 5 Distribution of wave number spectrum D_1 for the uniform cylinder.

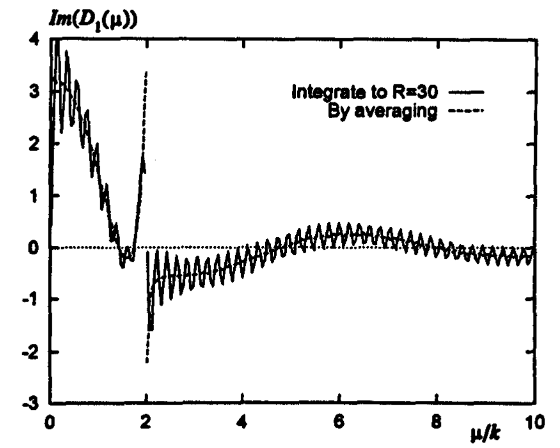

Fig. 6 Distribution of wave number spectrum D_1 for the uniform cylinder.