Some mathematical difficulties connected with asymptotic expansions

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1 Introduction

The Cauchy-Poisson Problem has been studied at these Workshops for many years. In particular we have been interested in the behaviour near the wavefront in the axisymmetric problem, and we only recently succeeded in finding a uniformly asymptotic expansion (for large non-dimensional times) which joined up smoothly with the stationary-phase expansion behind the wavefront, see [CNU 1995]. These expansions give a clear qualitative picture of the behaviour near the wavefront, of the same kind as Fresnel's treatment of a shadow boundary in optics, and Airy's treatment of a caustic. Since then Newman has followed up this theoretical treatment by unpublished numerical studies, in which he has compared the relative accuracy of the stationary-phase and uniform Airy expansions. These studies have shown that these asymptotic expansions do not agree closely unless the "large parameter" is indeed large. Which of the two is then correct, or are they both inaccurate ? It is of interest to examine this question theoretically, because the Cauchy-Poisson problem is typical of many problems in ship hydrodynamics, e.g. the Kelvin ship wave problem. In the present note we shall consider some of the characteristic defects of asymptotic expansions, and this is the purpose of the present note.

2 The Cauchy-Poisson problem

An instantaneous localized impulse acts on the free surface of water of finite constant depth h, the subsequent wave motion is to be found. In both two and three dimensions the solution for the velocity potential ϕ due to an initial impulse is easily found by the method of separation of variables. Here we shall consider only the two-dimensional problem, similar considerations apply in three dimensions. We find that

$$\phi(x,y,t) = \int_0^\infty \cos(t\{gk\tanh kh\}^{1/2})\frac{\cosh k(h-y)}{\cosh kh}\cos kx\,dk,$$

in an obvious notation, from which it follows that on the free surface

$$\phi(x,0,t) = \int_0^\infty \cos(t\{gk\tanh kh\}^{1/2})\cos kx\,dk.$$

By introducing non-dimensional parameters $X = x/h$ and $T = t(g/h)^{1/2}$ we see that we need to find integrals of the form

$$\int_{-\infty}^{\infty} \exp\left(iT\{(k\tanh k)^{1/2} \pm ak\}\right)dk,$$

where $a = X/T$. We shall confine attention to the integral

$$\int_{-\infty}^{\infty} \exp\left(iT\{(k\tanh k)^{1/2} - ak\}\right)dk.$$

or more precisely the convergent integral

$$\int_{\infty \exp(5\pi i/4)}^{\infty \exp(-\pi i/4)} \exp\left(iT\{(k \tanh k)^{1/2} - ak\}\right) dk.$$

which is the dominant integral when T is large, as is easily seen. This is a function of the two parameters T and a, and for small T and a it can be computed by direct numerical integration. As T increases the integrand oscillates more and more rapidly, and an ever-increasing number of significant figures needs to be retained in the numerical integration. We know that asymptotic expansions are specially appropriate when one of the parameters is large. When there is a second parameter a, as in our problem, then the asymptotic expansion typically involves functions of a single variable which is a function of T and a. (In our problem these functions are either circular functions or Airy functions, multiplied by simple decay factors.) This suggests a question, not considered here: Given an integral involving two parameters, when one of the parameters becomes large, does an asymptotic expansion necessarily exist ? (Probably not.)

In our problem the integrand is an exponential, for which the appropriate method is the Method of Stationary Phase or its complex-variable version, the Method of Steepest Descents. The phase is

$$\Phi(k, a) = (k \tanh k)^{1/2} - ak,$$

an odd function of k . The phase is stationary when $\partial\Phi/\partial k = 0$, an equation with a pair of equal and opposite roots $\pm k_0(a)$. When $a = 1$, (i.e. at the *wavefront* .) these roots coincide at $k = 0$; when $0 < a < 1$, (behind the wavefront) the roots are real and give rise to sine and cosine terms; when $1 < a < \infty$,(ahead of the wavefront) the roots are pure imaginary, only one root is relevant and gives rise to an exponential decay. Near the wavefront the two saddle points are nearly coincident, and the asymptotic expansion then involves Airy functions instead of exponentials. Such an expansion can be chosen so as to be *uniform*, i.e. so as to join up smoothly with the stationary-phase expansion: see [CFU 1957] . As we have already noted, the numerical precision of these asymptotic expansions is not adequate unless the "large parameter" T is indeed large.

3 Difficulties with asymptotic expansions

FIRST DIFFICULTY. In a convergent series the accuracy can in principle be increased indefinitely by taking more and more terms. In a non-convergent asymptotic series successive terms initially decrease to a smallest term (depending on T), then increase. Once the smallest term is reached the accuracy cannot be increased further by taking more terms. Even when the series is carried as far as its smallest terms the accuracy may not be good unless T is quite large.

SECOND DIFFICULTY. In our problems the coefficients (which are functions of a) are complicated and are not easily found.

THIRD DIFFICULTY. In the region of overlap both the stationary-phase expansion and the Airy expansion are valid. Which is more accurate ?

This poses the question: Why do successive terms in our expansions decrease so slowly, and what can be done to obtain better expansions ? What determines the rate of decrease of successive terms ?

Consider the simplest case of the Method of Steepest Descents, i.e. Watson's Lemma, the expansion of an integral of the form

$$\int_0^\infty g(k)\exp(-Tk)dk$$

for large T .To obtain the asymptotic expansion, we expand $g(k)$ in a power series,

$$g(k) = \sum_0^\infty g_m k^m,$$

and obtain

$$\int_0^\infty g(k)\exp(-Tk)dk \sim \sum_0^\infty g_m \frac{m!}{T^{m+1}}.$$

The rate of decrease of successive terms is thus seen to depend on the rate of decrease of the sequence of coefficients $\{g_m\}$, and it is well known that this depends on the distance R of the nearest singularity of $g(k)$ from the origin $k = 0$ in the complex k-plane. It follows that g_m is roughly of order R^{-m}. For Watson's Lemma we can find the precise remainder after M terms in the form of a double integral, but this is not necessarily useful because, as we have already seen, the remainder in an asymptotic series cannot be decreased indefinitely by increasing the number of terms. It would help if we could subtract out the contributions of the nearest singularities, but this has never yet been done successfully.

FOURTH DIFFICULTY. These difficulties relate equally to stationary-phase and Airy expansions, for there can be little doubt that the coefficients of the uniform Airy-type expansion have similar properties, with the additional complication that successive coefficients are not now explicit from a Taylor series but are determined by a Bleistein-sequence.

FIFTH DIFFICULTY. Proof of the validity of the Airy-type expansion. The present proof [Ursell 1972] is applicable only under assumptions which are not satisfied in our present problem. We confine ourselves to the uniform Airy expansion which joins up smoothly with the stationary-phase expansion. As is well known, this is found by introducing a new implicit variable of integration u such that

$$\Phi(k, a) = \frac{1}{3}u^3 - \alpha u,$$

where α is a certain analytic function of $1 - a$. Our integral is thus transformed into the canonical form

$$\int_{\infty\exp(5\pi i/6)}^{\infty\exp(\pi i/6)} G_0(u, \alpha)\exp\left(iT(\frac{1}{3}u^3 - \alpha u)\right)du,$$

where α is small near the wave front. In this integral we have

$$G_0(u, \alpha) = \partial k/\partial u.$$

a very complicated implicit function which needs to be expanded in a Bleistein sequence. This function is found to have four branch points not far from the origin $u = 0$, with a resultant slow decrease of successive terms. On the other hand, the standard proof of the validity of the Airy expansion assumes that the coefficient function $G_0(u, \alpha)$ is analytic everywhere. How can this standard proof be modified ? Unlike the other difficulties, previously listed, this might be capable of solution.

4 Conclusion

We have seen that our difficulties with asymptotic expansions of integrals occur because of singularities in Watson's Lemma.(often due to additional complex points of stationary phase in the k-plane not far from $k = 0$.) or equivalently the occurrence of singularities not far from $u = 0$ in the u-plane . We do not yet know how to deal with these effectively. The obvious treatment , when the "large parameter" T is only moderate, is by very accurate methods of direct quadrature. Another method is through the ideas of hyperasymptotics, i.e. the summation of terms beyond the smallest term in the asymptotic series. Little progress has so far been made. While the difficulties are apparently concerned with questions of pure-mathematical techniques, asymptotic expansions nevertheless give much useful information about physical behaviour. Thus the notion of group velocity originated from the mathematics of stationary phase. The Airy function which defines the development of the wave motion near the wavefront originated in the treatment of the caustic in optics.

References

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