Higher-order wave drift forces on bodies with a small forward speed based on a long wave approximation

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1 Introduction

The mean wave force acting on bodies, stationary or moving with a small forward speed in a wave field is considered. This force, the so called wave drift force, has shown to be of great importance within offshore technology. The change in the drift force because of the small speed, the wave drift damping, may be an important damping mechanism.

Calculation of the wave drift forces has traditionally been based on linear theory giving the drift force consistently to second order in the wave amplitude, the mean second order wave force. We here refer to Grue & Biberg (1993), who extended the theory to include a finite depth. In this work we use a long wave approximation to calculate higher order wave drift forces on a vertical cylinder in shallow water, but of interest is also the time-dependent higher order wave force. The latter is among others also considered by Jiang & Wang (1995), for stationary bodies. As a model we use one version of the weakly nonlinear and dispersive Boussinesq equations, see. e.g. Wu (1981), Pedersen (1989). We remark that the Boussinesq equations contain the fully nonlinear hydrostatic equations. The equation set is then modified to include a small current. It is necessary to point out that in many practical problems, the water depth is outside the limit of the long wave approximation. One of the intentions with the present work is however to indicate higher order effects on the wave drift force.

The body is exposed to incoming cnoidal waves, and the wave field around the body is solved numerically in space and time by the finite element method. Then the drift force is computed by first integrating the pressure over the body surface, and then time-averaging the periodic force. The wave drift damping is calculated by numerical differentiation of the drift force with respect to the small current.

2 Mathematical formulation

The problem is considered in a frame of reference (x, y, z) moving with the body, in which there is a small constant current U_0 in the positive x-direction. Assuming potential theory, the velocity field may be expressed by a velocity potential $\Phi(x, y, z, t)$, where t is the time. According to the long wave approximation used here, we then introduce a depth average velocity potential $\psi(x, y, t)$ by

$$\psi(x, y, t) = \frac{1}{h + \eta} \int_{-h}^{\eta} \Phi(x, y, z, t) dz \tag{1}$$

Here $\eta(x, y, t)$ denotes the surface elevation, and the constant h is the mean water depth. We observe that the unknowns ψ and η are only functions of the horizontal coordinates. Furthermore ψ is divided into two parts $\psi(x, y, t) = \phi(x, y, t) + \phi_0(x, y)$ where ϕ and ϕ_0 represent the velocity potential due to the waves and the small current respectively. Typical wave length λ_0, and typical wave height H_0 are then defined, and three important dimensionless parameters α_0, α and ϵ given by

$$\alpha_0 = \frac{U_0}{\sqrt{gh}}, \qquad \alpha = \frac{H_0}{h}, \qquad \epsilon = \frac{h^2}{\lambda_0^2} \tag{2}$$

will be present in the model. The gravitational acceleration is denoted by g. The actual version of the Boussinesq equations that will be used, and modified to include a small current, has the advantage of containing the unknown velocity potential instead of the two horizontal velocity-components. This reduces the total number of unknowns. The equation set for the unknown potentials ϕ_0 and ϕ, and the surface elevation η is

$$\nabla^2 \phi_0 = 0 \tag{3}$$

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}(\nabla\phi)^2 - \frac{h^2}{3}\nabla^2\frac{\partial \phi}{\partial t} + \nabla\phi_0 \cdot \nabla\phi = 0 \tag{4}$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot ((h+\eta)\nabla\phi) + \nabla\phi_0 \cdot \nabla\eta = 0 \tag{5}$$

where neglected terms are $O(\alpha_0^2, \alpha_0\alpha\epsilon, \alpha^2\epsilon, \alpha\epsilon^2)$. Eq. (4) represents conservation of momentum, while eq. (5) represents mass conservation.

The force acting on the body is obtained by integrating the pressure over the body surface. The depth integration is done analytically, and for the time dependent force $\mathbf{F}(t)$ we then obtain the following expression in terms of ϕ_0, ϕ and η

$$\mathbf{F}(t) = \rho h \int_{\Gamma_B} \left(-\frac{\partial \phi}{\partial t} + \frac{g}{2h}\eta^2 - \frac{1}{2}(\nabla\phi)^2 - \nabla\phi_0 \cdot \nabla\phi\right)\mathbf{n}\,d\Gamma \tag{6}$$

Here Γ_B denotes the contour line of the body, $\mathbf{n}$ is the normal vector pointing out of the fluid domain and ρ is the fluid density. By time-averaging the force with respect to the wave period, we obtain the drift force $\overline{\mathbf{F}}$

$$\overline{\mathbf{F}} = \rho h \int_{\Gamma_B} \overline{\left(\frac{g}{2h}\eta^2 - \frac{1}{2}(\nabla\phi)^2 - \nabla\phi_0 \cdot \nabla\phi - \frac{1}{6}\left(\frac{\partial \eta}{\partial t}\right)^2\right)\mathbf{n}\,d\Gamma} \tag{7}$$

In this expression neglected terms are $O(\alpha_0^2, \alpha_0\alpha^2\epsilon, \alpha^4\epsilon, \alpha^2\epsilon^2)$, which is consistent with (3) - (5). The wave drift force is then expanded in order of α_0 by $\overline{\mathbf{F}} = \mathbf{F}_0 + \alpha_0\mathbf{F}_1$, where $\mathbf{F}_0$ is the zero speed drift force, and $\alpha_0\mathbf{F}_1$ is the wave drift damping force.

3 Numerical simulation

The numerical solution is performed by using the finite element method, with the ability of easily consider bodies of arbitrary shapes. Differentiation with respect to time is approximated by finite difference. For further details about the numerical method, we refer to Irmann-Jacobsen (1989) where (4)-(5) have been solved numerically when $\phi_0 \equiv 0$.

The model is applicable to an arbitrary fluid domain, but in the present study we want to calculate the drift force on a body in an unbounded fluid, with the incident wave field propagating in positive x-direction. We therefore define the simulating area as a square basin, (see. Fig 1), and solve (3)-(5) with the following initial and boundary conditions.

Eq. (3) for the unknown ϕ_0:

$$\frac{\partial \phi_0}{\partial n} = -U_0, U_0 \quad \text{on} \quad \Gamma_L, \Gamma_R \tag{8}$$

$$\frac{\partial \phi_0}{\partial n} = 0 \quad \text{on} \quad \Gamma_S, \Gamma_B \tag{9}$$

Eq. (4) and (5) for the unknowns ϕ and η:

$$\phi = \eta = 0 \quad t = 0 \tag{10}$$

$$\phi = \phi_I(t), \qquad \eta = \eta_I(t) \quad \text{on} \quad \Gamma_L \tag{11}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \Gamma_B, \Gamma_S, \Gamma_R \tag{12}$$

Considering the solution of (4) and (5), (11) is the essential and (12) the natural boundary condition, the latter being the rigid wall condition. The incident waves given by η_I and ϕ_I with given wave length λ, and given wave height H, are the cnoidal wave solution of (4)-(5).

It is necessary to discuss some aspects about the discretization and the choice of boundary conditions. The time-averaging of the force must not be done before the wave field around the body has become nearly periodic in time. We must therefore either impose a radiation condition, or use a very large simulating area. The problem with the first is that it is difficult to ensure that the boundary does not reflect any significant waves, this has been the outcome from simulations where a radiation condition has been applied. In the latter case, one normally need a large number of elements. It is however found, by simulation of solitary waves propagating in one direction, that by increasing gradually and not to fast the element size, reflection because of grid-variation may bee neglected. We therefore use a large basin, with increasing element sizes in the outgoing region (i.e. downstream and to the side of the body see. Fig 1). It is then like wise to use the rigid wall condition on Γ_S and Γ_R. A time-averaging procedure is then established, and the drift force may be computed within a reasonable CPU time. An analytical solution of the second order drift force based on (4)-(5) when $\phi_0 \equiv 0$, has been developed for a circular cylinder. The mean second order wave force has then been computed numerically and convergence-tested with the analytical solution, with very good accuracy.

4 Results

In the first example, the body is a circular cylinder, with radius $R = 5h$, the size of the basin is $110h \times 110h$ with 11745 elements in half of the fluid domain. The body is exposed to an incident cnoidal wave train, and Fig. 2a shows an example of the x-component of $\mathbf{F}(t)$ at two different values of U_0. Fig. 2b and Fig. 2c then shows the x-component of the zero speed drift force and the wave drift damping at different values of the wave height, as a function of the wave length. The numerical differentiation of the drift force is done about $U_0 = 0.0$ with $\Delta U_0 = 0.04\sqrt{gh}$. $H = 0.0$ means second order theory. What is interesting to note is that both the wave drift coefficient $\frac{F_{x0}}{\rho g H^2 2R}$ and the wave drift damping coefficient $\frac{F_{x1}}{\rho g H^2 2R}$ are decreasing with increasing values of $\frac{H}{h}$. In the last example, Fig. 2d, the body is a model of a ship with length $L = 10h$ and beam $B = 1.79h$. In this case we see that the wave drift damping coefficient is not always less for the steepest waves.

References

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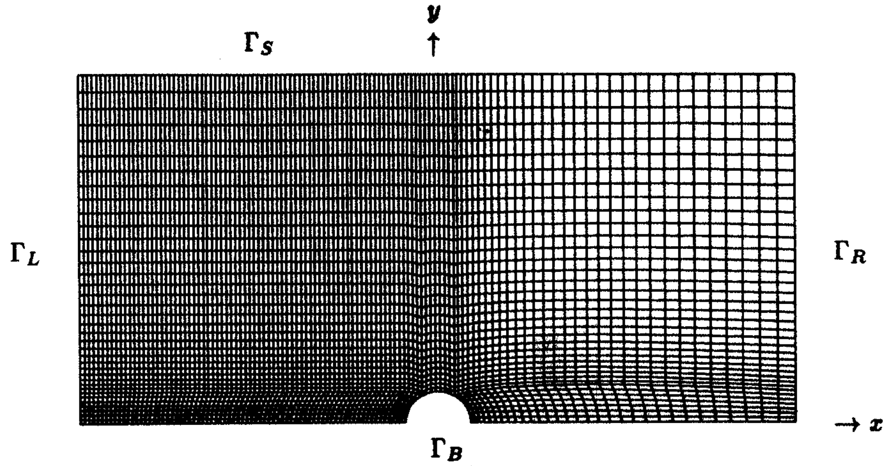

Figure 1: Discretization of half of the fluid domain, for the body being a circular cylinder. 4095 elements.

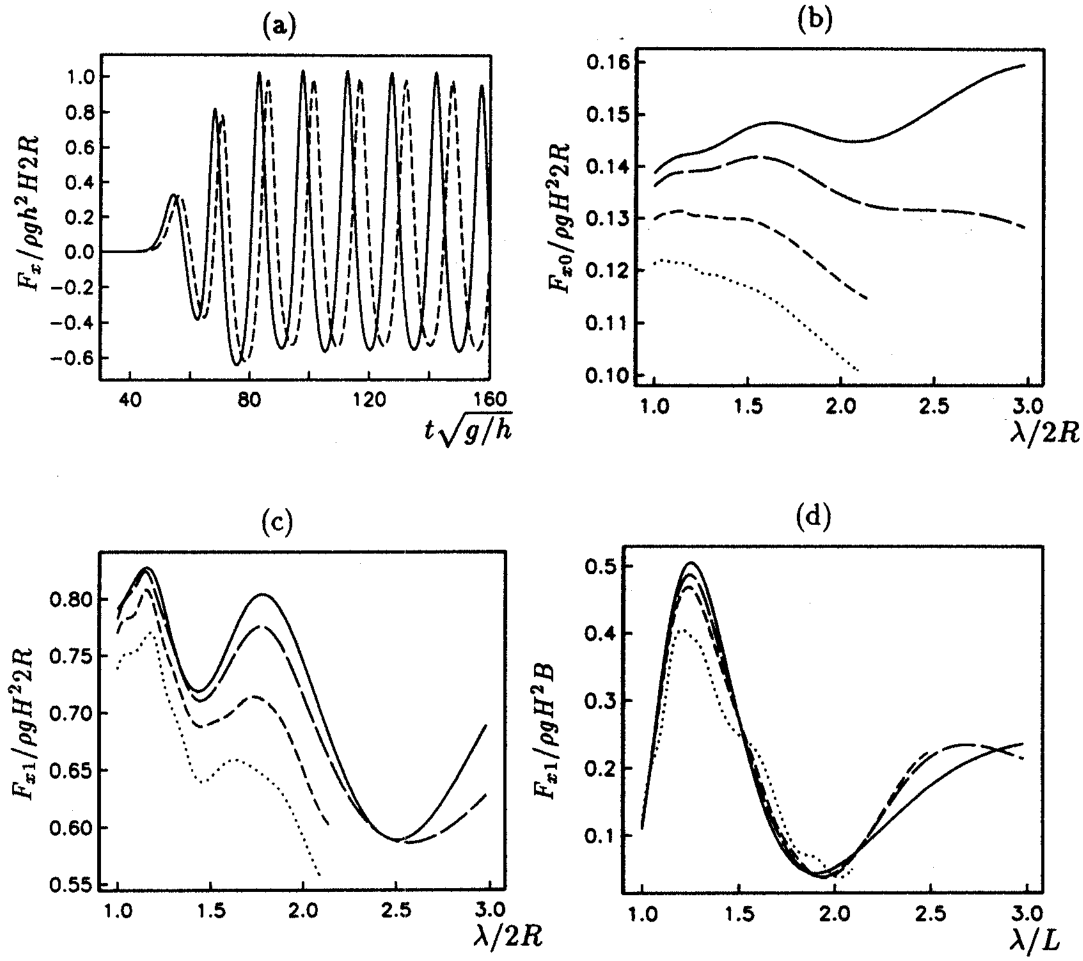

Figure 2: (a): The x-component of $\mathbf{F}(t)$ for the circular cylinder, $H/h = 0.2$, $\lambda/2R = 1.5$, $U_0/\sqrt{gh} =$ 0.02 (solid line) and -0.02 (dashed line). (b) and (c): The x-component of $\mathbf{F}_0$ and $\mathbf{F}_1$ for the circular cylinder, $H/h = 0.0$ (solid line), 0.1 (long dashed line), 0.2 (dashed line) and 0.3 (dotted line). (d): F_{x1} for the ship, $H/h = 0.0$ (solid line), 0.1 (long dashed line), 0.15 (dashed line) and 0.3 (dotted line).