Numerical investigations into non-uniqueness in the two-dimensional water-wave problem

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Introduction

The proof, by McIver (1996), of the existence of eigenvalues (trapped modes) in the two-dimensional water-wave problem has answered the general question of uniqueness in the negative but has created many new and interesting questions for investigation. For example, given a particular geometry for which there is no uniqueness proof, we have no general method for determining whether or not eigenvalues exist.

In this work we will consider a class of geometries, namely pairs of symmetrically-placed surface-piercing angled barriers in water of infinite depth, and use an integral equation approach to investigate the existence or otherwise of eigenvalues. This particular class of geometries has been chosen for three reasons. First the problem is not known to be unique, though recent results of N. Kuznetsov (Linton and Kuznetsov 1997) provide ranges of the frequency parameter for which uniqueness is assured. Secondly it is sensible to choose a geometry which shares as many of the characteristics of those computed numerically by McIver (1996) as possible, and her results were for pairs of surface-piercing bodies. Thirdly we must choose a geometry for which mathematical progress can be made. The problem of wave scattering by a single surface-piercing angled barrier has been solved using hypersingular integral equations by Parsons and Martin (1994) and it is their approach that has been followed here.

Formulation

We consider the case of an inclined surface-piercing barrier next to a vertical wall. By symmetry any eigenvalues for this problem will correspond to trapped modes for a pair of symmetrically-placed surface-piercing barriers. The geometry is illustrated in Figure 1. Following Parsons and Martin (1994) we set the problem up as a hypersingular integral equation. Trapped modes then correspond to non-trivial velocity potentials ϕ for which the discontinuity across the plate, $[\phi]$, satisfies

$$\oint_\Gamma [\phi(q)] \frac{\partial^2 G(p,q)}{\partial n_p \partial n_q} \mathrm{d}s_q = 0, \qquad p \in \Gamma,$$

where p and q are points on the plate, Γ, the integral is a Hadamard finite-part integral and $G(P,Q)$ is the standard free-surface Green's function. The free-surface boundary condition satisfied by G is $KG + \partial G/\partial y = 0$ on $y = 0$ and if a trapped mode exists for a particular Ka ($\equiv \omega^2 a/g$) then Ka is the corresponding eigenvalue.

The unknown function $[\phi]$ is approximated as a finite sum of Chebyshev polynomials of the second kind leading to an equation of the form

$$\sum_{n=1}^{N} A_{mn}(Ka, b/a, \theta)c_n = 0, \qquad m = 1,\dots,N.$$

Eigenvalues correspond to values of Ka for which the determinant of the matrix $\boldsymbol{A}$ with elements A_{mn}, $m,n = 1,\dots,N$, vanishes in the limit as $N \to \infty$. It is clear that with

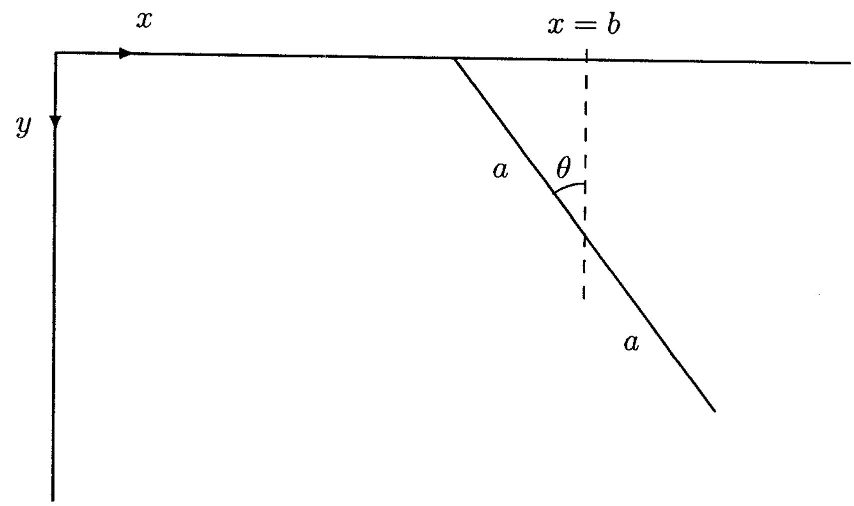

Figure 1: Definition sketch

this approach we are unlikely to be able to *prove* the existence or otherwise of eigenvalues. However we are able to provide strong numerical evidence for the existence of eigenvalues and also to show that if the eigenvalues do indeed exist then they are unstable to perturbations in the geometry in a sense which will be made precise. Even if eigenvalues do not exist, we can show that there are values of Ka in the vicinity of which a related scattering problem has qualitatively different behaviour from that at more typical values.

Results and discussion

A number of results will be presented, both illustrating the powerful evidence for existence of eigenvalues and computations of the eigenvalues themselves. A key point is that the elements of $\boldsymbol{A}$ are complex and so, in general, is its determinant. One way of looking at the problem is to find zeros of the real and imaginary parts of the determinant, which is numerically straightforward as each is a real-valued function which passes through zero, and then to see if these zeros occur at the same point.

For a particular example we fix the inclination of the barrier at $\theta = \pi/4$. The values of Ka at which zeros of the real and imaginary parts of the determinant occur are plotted in Figure 2 for a range of values of b/a. The solid line represents a zero of the real part and the dotted lines represent zeros of the imaginary part. We see that the real part of the determinant has a zero for all values of b/a in the range shown in the figure whereas the imaginary part has a pair of zeros which coalesce and disappear near $b/a = 2.21$. An eigenvalue exists if the lower dotted line actually touches the solid line. The qualitative behaviour shown in the figure is typical.

Numerical results suggest that eigenvalues, if they exist, are *unstable* in the sense that it is possible that an arbitrarily small change to the geometry will cause the eigenvalue to be lost. This is in contrast to previously discovered trapped-mode phenomena in water waves. For example, in the case of a submerged horizontal cylinder, edge waves exist whatever the cross-section and so the geometry can be changed arbitrarily without the eigenvalue disappearing. However in the problem considered here arbitrarily small perturbations to the geometry can lead to the disappearance of an eigenvalue. It is possible, however, to vary b/a and θ at the same time in such a way as to retain the eigenvalue. In other words the numerical evidence suggests that eigenvalues exist on curves in $(b/a, \theta)$ space.

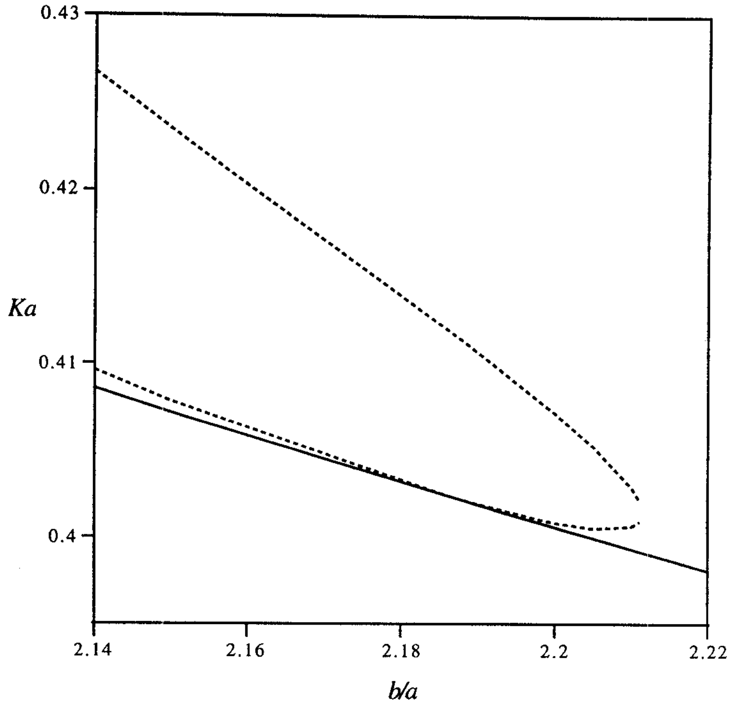

Figure 2: Values of Ka at which the real part (—) and the imaginary part $(\cdots)$ of the determinant vanish.

Simple continuity considerations show that the unstable nature of the eigenvalues is related to the tangential nature of the contact in Figure 2 and to the need to consider a complex-valued determinant rather than a real one (which would be more convenient). This is illustrated schematically in Figure 3. In diagrams a) and b), which are to be compared with Figure 2, the solid curves represent the value of Ka at which the real part of the determinant vanishes and the dotted curves the value at which the imaginary part vanishes. Diagram b) (top) shows the situation as we find it in Figure 2 whereas diagram a) (top) shows the situation where the curves intersect at a non-zero angle. In the lower figures we see the possible effect of a small perturbation. The fact that we observe behaviour like that in case b) suggests that arbitrarily small perturbations exist which totally destroy the eigenvalues.

Diagram c) represents the situation where we are looking for a zero of a real-valued determinant. The horizontal line represents zero and the curve is the value of the determinant. Here again the figures illustrate that provided the determinant passes through zero, sufficiently small perturbations will not destroy the eigenvalue. In our problem this is not the case and diagram d) shows why finding the zero of a complex-valued determinant is then appropriate. The curve again represents the value of the determinant, now plotted in the complex plane, with Ka varying as we move along the curve. In this case the determinant can be displaced from the origin by an arbitrarily small perturbation.

We can also shed light on the situation by considering a circular cylinder on the centreline of a two-dimensional parallel-plate waveguide, as considered by Callan, Linton, and Evans (1991) who showed that trapped modes, antisymmetric about the centreline of the guide, exist for such a geometry. If we impose the antisymmetry the problem reduces to one of finding the zeros of a real determinant and the eigenvalues are stable (in the sense we have used the word in this paper), always remembering that we cannot consider perturbations which

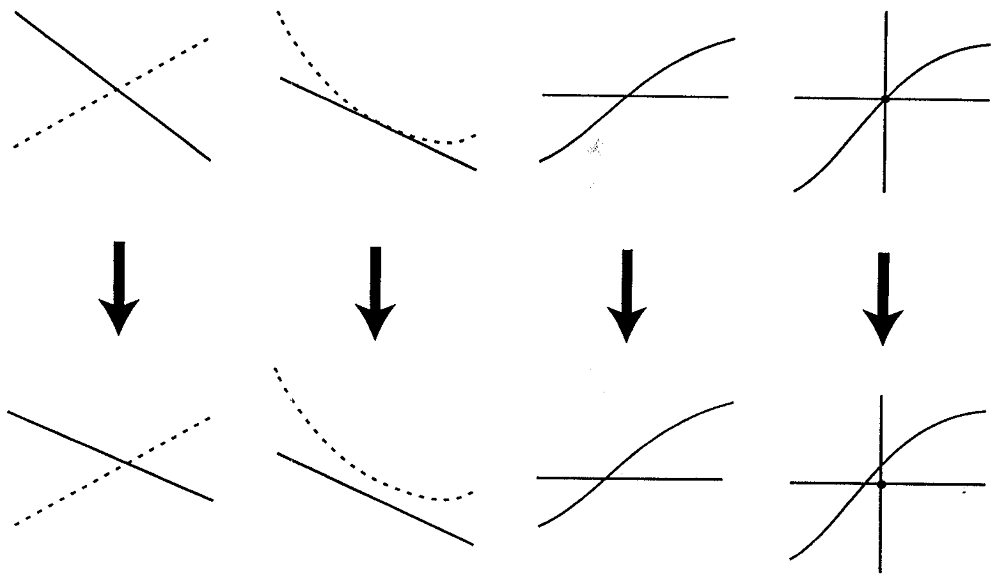

Figure 3: Unstable nature of eigenvalues. See text for details.

destroy the symmetry. This is what we would expect since Evans, Levitin, and Vassiliev (1994) have shown that trapped modes exist for bodies of any shape (subject to some mild restrictions) on the centreline of a waveguide provided the geometry is symmetric about this line. However if we make no assumptions about symmetry, thus allowing us to consider arbitrary perturbations in the geometry, we find that we must consider a complex-valued determinant. Again this is expected since an arbitrarily small displacement of the cylinder from the centreline (a perturbation which destroys the symmetry) causes the eigenvalue to disappear.

References

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DISCUSSION

Tuck E.O.: Your instability argument is very interesting. However, it is possible that small perturbations might change the "touching" phenomenon in the opposite direction. That is, instead of eliminating the non-uniqueness at a particular wavenumber $k = k_o$, it might create non-uniqueness at two neighbouring frequencies $k = k_o \pm \varepsilon$.

Linton C.M.: Whilst this is true the fact that the eigenvalue also corresponds to a complex-valued function passing through zero suggests that a transition to zero eigenvalues rather than to two is more likely.
Also in other fields, notably quantum mechanics, eigenvalues embedded in the continous spectrum (as those considered in this paper are) are known to be unstable to small perturbations.