Higher order wave diffraction of water waves by an array of vertical circular cylinders

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It is well known that the wave forces exerted on a multicolumn offshore structure (TLP, semi-sub, GBS, ...) can be seriously affected by the effects of interaction between cylinders. The calculation of forces by simply summing the forces for isolated cylinders is usually wrong. As far as the linear theory is considered, there exist today numerous numerical models which calculate these interaction effects correctly. However, at second order there is only few of them and the calculations involved are very time consuming. That is the reason why some authors try to seek for semianalytical solutions, in idealized configurations, which are considerably less expensive and more precise. At first order, this was done by different authors but the most complete methodology was presented in [2]. At second order there are two approachs recently proposed [3,5]. Even if these two methods are similar there are some important differences in the methodology and in the numerical implementation. The purpose of this paper is to discuss more in detail the method proposed in [3] and to give some clarifications about the numerical implementation which has to be adopted in order to support eventual third order calculations.

General methodology

The main difficulty associated with higher order problems is the treatement of the free surface integral which is involved in the solution. The free surface condition, which is the main difficulty of the problem, at higher order is given by :

$$-\alpha\psi+\frac{\partial\psi}{\partial z}=Q \tag{1}$$

and use of any method involves the evaluation of an integral over the entire free surface in order to take account for the forcing term Q. The method that we use to solve this problem is the combination of the Linton and Evans method [2] for first order diffraction for an array of vertical cylinders, and the semi-analytical solution for the second order diffraction by a single cylinder [1,4].
All notations correspond to the following configuration :

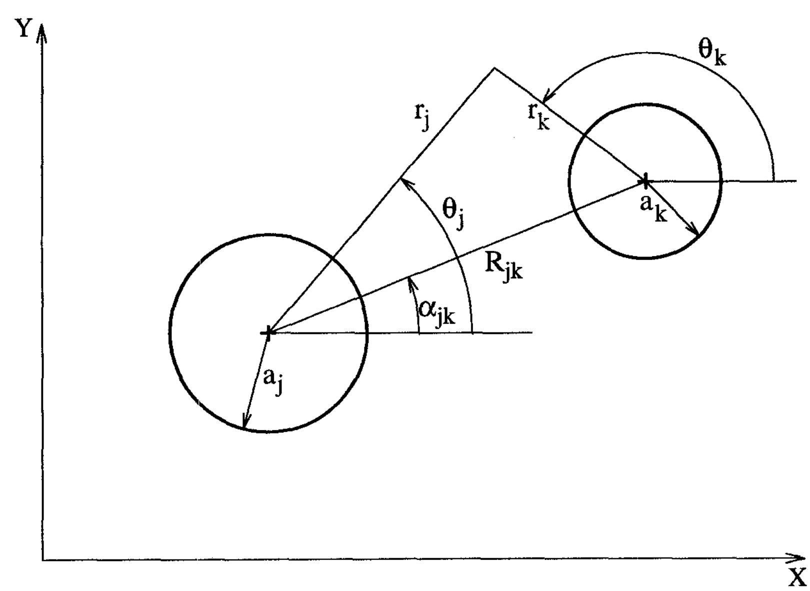

Figure 1. Basic configuration.

The well known Green's identity for one point outside of the fluid domain, can be written (for the sake of clarity we consider homogeneous Neuman boundary condition on the body) :

$$-\iint_{S_{B0}} \psi \frac{\partial G}{\partial n}\, dS = -\iint_{S_F} G\, Q\, dS \tag{2}$$

where ψ is the higher order potential, Q is its forcing term on the free surface and G is the Green function which, in the coordinate system of the k-th cylinder, can be written as :

$$G(r_k,\theta_k,z_k;\rho_k,\upsilon_k,\zeta_k) = \sum_{m=-\infty}^{\infty} \left\{ -\frac{i}{2} C_0 \begin{bmatrix} H_m(\kappa_0 r_k) J_m(\kappa_0 \rho_k) \\ J_m(\kappa_0 r_k) H_m(\kappa_0 \rho_k) \end{bmatrix} f_0(z) f_0(\zeta) \right.$$

$$\left. -\frac{1}{\pi}\sum_{n=1}^{\infty} C_n \begin{bmatrix} K_m(\kappa_n r_k) I_m(\kappa_n \rho_k) \\ I_m(\kappa_n r_k) K_m(\kappa_n \rho_k) \end{bmatrix} f_n(z) f_n(\zeta) \right\} e^{im(\theta_k-\upsilon_k)} \quad ; \quad \begin{bmatrix} r_k>\rho_k \\ r_k<\rho_k \end{bmatrix} \tag{3}$$

where $\kappa_0 \tanh\kappa_0 H = -\kappa_n \tan\kappa_n H = \alpha$, H being the water depth, and :

$$f_0(z)=\frac{\cosh\kappa_0(z+H)}{\cosh\kappa_0 H} \quad ; \quad f_n(z)=\frac{\cos\kappa_n(z+H)}{\cos\kappa_n H} \quad ; \quad C_0=[2\int_{-H}^{0} f_0^2(z)dz]^{-1} \quad ; \quad C_n=[2\int_{-H}^{0} f_n^2(z)dz]^{-1} \tag{4}$$

We develop now the potential on the k-th cylinder in the eigenfunction expansion as follows :

$$\psi^k = \sum_{m=-\infty}^{\infty} [B^k_{m0} f_0(\zeta) + \sum_{n=1}^{\infty} B^k_{mn} f_n(\zeta)]\, e^{im\upsilon_k} \tag{5}$$

After writing the equation (2) for one point inside the cylinder $k \rightarrow (r_k=a_k-\delta\ ,\ 0<\delta\leq a_k)$, carrying out the integration by ζ, using the orthogonality of the functions $f_n(z)$, using the Graff's addition theorem for Bessel functions, exploiting the orthogonality of the functions $e^{im\theta}$ and rearranging the different terms we obtain for the part associated with the B^k_{m0} coefficients :

$$B^k_{m0}+\sum_{j\neq k}\frac{a_j}{a_k}\sum_{n=-\infty}^{\infty} B^j_{n0}\frac{J'_n(\kappa_0 a_j)}{H'_m(\kappa_0 a_k)} H_{n-m}(\kappa_0 R_{jk}) e^{i(n-m)\alpha_{jk}}$$

$$=-\frac{C_0}{\pi a_k \kappa_0 H'_m(\kappa_0 a_k)}\int\int_{S_F} H_m(\kappa_0\rho_k) e^{-im\upsilon_k} Q(\rho_k,\upsilon_k) dS \quad , \quad k=1,N \quad ; \quad m=-\infty,\infty \tag{6}$$

This expression represent the final system of equations for the unknown coefficients B^k_{m0}.
Similarly the expression for the coefficients B^k_{mn} can be obtained :

$$B^k_{mn}+\sum_{j\neq k}\frac{a_j}{a_k}\sum_{l=-\infty}^{\infty} B^j_{ln}\frac{I'_l(\kappa_n a_j)}{K'_m(\kappa_n a_k)}(-1)^l K_{l-m}(\kappa_n R_{jk}) e^{i(l-m)(\alpha_{jk}-\pi)}$$

$$=-\frac{C_n}{\pi a_k \kappa_n K'_m(\kappa_n a_k)}\int\int_{S_F} K_m(\kappa_n\rho_k) e^{-im\upsilon_k} Q(\rho_k,\upsilon_k) dS \quad , \quad k=1,N \quad ; \quad m=-\infty,\infty \tag{7}$$

Second order diffraction

The main difficulty in solving the above equations is the evaluation of the free surface integrals which should exclude the cylinder surfaces. In order to minimize the 2D integration the following procedure can be adopted for 4 equaly spaced cylinders :

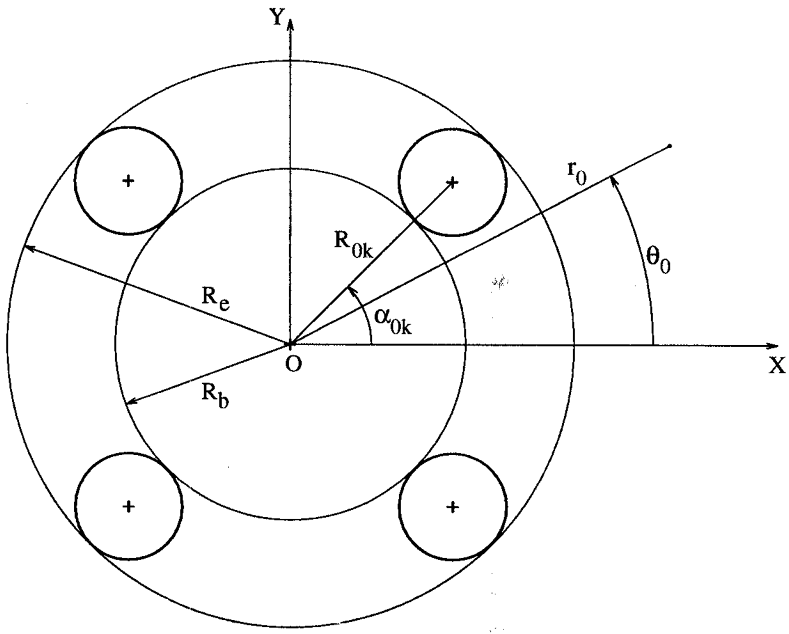

Figure 2. Differents regions for integration.

We recall the first order solution [2] :

$$\phi_I = -\frac{igA}{\omega} f_0(z) \sum_{m=-\infty}^{\infty} e^{im(\pi/2-\beta)} J_m(k_0 r_0) e^{im\theta_0} \quad ; \quad \phi_D = f_0(z) \sum_{k=1}^{N} \sum_{m=-\infty}^{\infty} A_m^k Z_m^k H_m(k_0 r_k) e^{im\theta_k} \tag{8}$$

and the expression for the forcing term Q in the second order diffraction problem :

$$Q_D^{(2)} = \frac{i\omega}{2g}(3\nu^2 - k_0^2)(\phi_D\phi_D + 2\phi_I\phi_D) + \frac{i\omega}{g}(\nabla_0\phi_D\nabla_0\phi_D + 2\nabla_0\phi_I\nabla_0\phi_D) \tag{9}$$

Using the Graff's theorem we can write the first order diffraction potential ϕ_D in the global coordinate system (X,Y) i.e. in terms of (r_0,θ_0) :

$$\phi_D = f_0(z) \sum_{m=-\infty}^{\infty} \left\{ \sum_{k=1}^{N} \sum_{n=-\infty}^{\infty} A_n^k Z_n^k H_{n-m}(k_0 R_{0k}) e^{i(n-m)(\alpha_{0k}-\pi)} \right\} J_m(k_0 r_0) e^{im\theta_0} \qquad r_0 \le R_b \tag{10}$$

$$\phi_D = f_0(z) \sum_{m=-\infty}^{\infty} \left\{ \sum_{k=1}^{N} \sum_{n=-\infty}^{\infty} A_n^k Z_n^k J_{n-m}(k_0 R_{0k}) e^{i(n-m)(\alpha_{0k}-\pi)} \right\} H_m(k_0 r_0) e^{im\theta_0} \qquad r_0 \ge R_e \tag{11}$$

This allows us to write the forcing term $Q_D^{(2)}$ in the form (in the regions $r_0 \le R_b$ and $r_0 \ge R_e$) :

$$Q_D^{(2)} = \sum_{m=-\infty}^{\infty} Q_{Dm}^{(2)}(r_0) e^{im\theta_0} \tag{12}$$

At the same time we write the term $H_m(\kappa_0\rho_k)e^{-im\upsilon_k}$ [eqn. (6)] in terms of (ρ_0,υ_0) [similar procedure apply for $K_m(\kappa_n\rho_k)e^{-im\upsilon_k}$ in eqn. (7)]:

$$H_m(\kappa_0\rho_k)e^{-im\upsilon_k} = \sum_{n=-\infty}^{\infty} H_{m-n}(\kappa_0 R_{0k}) e^{i(m-n)(\pi-\alpha_{0k})} J_n(\kappa_0\rho_0) e^{-in\upsilon_0} = \sum_{n=-\infty}^{\infty} \alpha_{mn}^k J_n(\kappa_0\rho_0) e^{-in\upsilon_0} \qquad \rho_0 < R_{0k} \tag{13}$$

$$H_m(\kappa_0\rho_k)e^{-im\upsilon_k} = \sum_{n=-\infty}^{\infty} J_{m-n}(\kappa_0 R_{0k}) e^{i(m-n)(\pi-\alpha_{0k})} H_n(\kappa_0\rho_0) e^{-in\upsilon_0} = \sum_{n=-\infty}^{\infty} \beta_{mn}^k H_n(\kappa_0\rho_0) e^{-in\upsilon_0} \qquad \rho_0 > R_{0k} \tag{14}$$

The free surface integral in (6) is now divided into three parts :

$$\int\int_{S_F} = \int_0^{2\pi}\int_0^{R_b} + \int\int_{S_d} + \int_0^{2\pi}\int_{R_e}^{\infty} \tag{15}$$

where S_d is the surface between R_b and R_e without cylinder surfaces.
In the first and third integral, the integration by υ can be carried out explicitely :

$$\int_0^{2\pi}\int_0^{R_b} H_m(\kappa_0\rho_k)e^{-im\upsilon_k} Q_D^{(2)}\rho_0 d\rho_0 d\upsilon_0 = 2\pi \sum_{n=-\infty}^{\infty} \alpha_{mn}^k \int_0^{R_b} J_n(\kappa_0\rho_0) Q_{Dn}^{(2)}(\rho_0)\rho_0 d\rho_0 \tag{16}$$

$$\int_0^{2\pi}\int_{R_e}^{\infty} H_m(\kappa_0\rho_k)e^{-im\upsilon_k} Q_D^{(2)}\rho_0 d\rho_0 d\upsilon_0 = 2\pi \sum_{n=-\infty}^{\infty} \beta_{mn}^k \int_{R_e}^{\infty} H_n(\kappa_0\rho_0) Q_{Dn}^{(2)}(\rho_0)\rho_0 d\rho_0 \tag{17}$$

In this way we reduced the 2D integration to 1D and, in the same time, the most difficult integral (from R_e to ∞) is put in the same form as in the single cylinder case and the same method can be used for its evaluation. So the only "real" 2D integral evaluation needed is over the surface S_d, and since this surface is relatively small, this integration can be done with little computational effort.
Knowing the second order potential on the surfaces of the cylinders, the second order potential at any point in the fluid can be calculated by using the Green's theorem :

$$\psi_D^{(2)} = \sum_{j=1}^{N} 2\pi a_j \sum_{m=-\infty}^{\infty} [B_{m0}^j \frac{i\kappa_0}{4} J_m'(\kappa_0 a_j) f_0(z) H_m(\kappa_0 r_j) + \sum_{n=1}^{\infty} B_{mn}^j \frac{\kappa_n}{2\pi} I_m'(\kappa_n a_j) f_n(z) K_m(\kappa_n r_0)] e^{im\theta_j} - \int\int_{S_F} G Q_D^{(2)} dS \tag{18}$$

The reduction to the global coordinate system (r_0,θ_0), in the free surface integral, can again be very useful.

What about third order ?

The task seems to be very difficult, but not hopeless. If we are interested only in the forces we can use the well known Haskind relations to avoid the explicit calculation of the third order potential :

$$F_{3j}^{(3)} = \int\int_{S_B} \psi_D^{(3)} N_j dS = -\int\int_{S_B} \frac{\partial \psi_I^{(3)}}{\partial n} dS + \int\int_{S_F} \phi_j Q_D^{(3)} dS \tag{19}$$

where $F_{3j}^{(3)}$ is the part of the third order forces induced by the third order diffraction potential $\psi_D^{(3)}$, $\psi_I^{(3)}$ is the third order incident potential, ϕ_j is the assisting radiation potential and $Q_D^{(3)}$ is the third order forcing term [4].
The main problem in the evaluation of this forces is the calculation of the free surface integral which requires the knowledge of the second order potential and some of its derivatives over the entire free surface. Even if the expression (18) can be used directly for the calculation of the potential $\psi_D^{(2)}$, some important numerical problems (treatement of the logaritmic singularity at the free surface, calculation of the derivatives, problems of convergence, ...) should be solved, and this is a rather complicated task.

References

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