On uniqueness and trapped modes in the water-wave problem for a surface-piercing axisymmetric body

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1 Introduction

The problem of uniqueness of the frequency-domain solution to the linearised water-wave problem is fundamental, as was recently highlighted by Ursell (**1**) who placed it first in his list of (at the time) unsolved problems. A number of uniqueness results have been established for specific geometries. In particular, John (**2**) proved the most widely known theorems for surface-piercing obstacles in two and three dimensions. Simon & Ursell (**3**) generalised John's 2D theorem to cover a wider class of obstacles including totally submerged ones. The first uniqueness theorem for a 2D obstacle separating a portion of the free surface from infinity was obtained by Kuznetsov (**4**). Simon & Kuznetsov (**5**) generalised this result to the case of a toroidal surface-piercing body. However, there is no general theorem giving necessary and sufficient criteria for either two- or three-dimensional geometries. The fact that some necessary conditions must be fulfilled for uniqueness follows from the recent achievement of M. McIver (**6**) who constructed the first examples of non-uniqueness for the two-dimensional water-wave problem. These 'trapped modes' were constructed from two equal-strength wave sources placed in the free surface and positioned so that the waves radiated to each infinity by one source are cancelled by the other. She proved that there exist families of streamline pairs surrounding the sources that can be interpreted as two surface-piercing structures. The corresponding three-dimensional problem was considered by McIver & McIver (**7**) who constructed solutions from a ring source with a vertical axis of symmetry placed in the free surface. The radius of the ring is chosen to eliminate the radiated wave; this results in a standing-wave motion that decays more quickly in the radial direction than any propagating wave solution. The stream surfaces of the flow correspond to toroidal structures floating in the free surface.

The present work extends the work of McIver & McIver (**7**) in two ways. First of all it is shown that, for certain toroidal structures, uniqueness for any mode of the fluid motion may be established over particular ranges of frequency. Secondly, trapped mode solutions are constructed numerically for non-axisymmetric modes in the presence of an axisymmetric structure.

2 Formulation

An inviscid, incompressible fluid occupies the half-space $y \geq 0$ with Cartesian coordinates (x, y, z) chosen so that $y = 0$ corresponds to the free surface and y is directed vertically downwards. Horizontal polar coordinates (r, θ) are defined by $x = r\cos\theta$, $z = r\sin\theta$. A structure, axisymmetric about the y axis and with submerged volume D and wetted surface S, floats in the free surface. The fluid domain is denoted by W and the free surface by F. The structure is toroidal in shape so that the free surface is in two distinct parts; the outer free surface is denoted by F_+ and the inner free surface of radius b is denoted by F_-. The geometry is sketched in Fig. 1.

Within the framework of the linearised theory, a time-independent potential ϕ corresponding to a trapped wave motion must satisfy

$$\nabla^2\phi = 0 \quad \text{in} \quad W, \tag{1}$$

the free-surface condition

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad F = F_- \cup F_+ \tag{2}$$

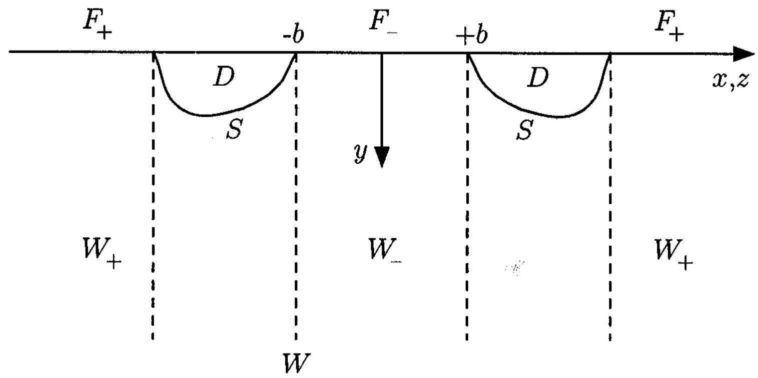

Figure 1: Sketch of geometry

and the body boundary condition

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad S, \tag{3}$$

where $\partial/\partial n$ indicates differentiation in a direction normal to the surface S of the structure. Here $K = \omega^2/g$, where g is the acceleration due to gravity. The radiation condition requires that ϕ and $\nabla\phi$ decay at infinity in such a way that the energy of a trapped fluid motion is finite and therefore

$$\int_W |\nabla\phi|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_F |\phi|^2 \,\mathrm{d}x\,\mathrm{d}z < \infty. \tag{4}$$

3 A uniqueness theorem

For modes of the form

$$\phi = \phi_n(r,\theta,y) = \phi^{(n)}(r,y)\cos n\theta, \quad n = 0,1,\dots, \tag{5}$$

energy arguments related to those used by John (**2**) and Simon & Ursell (**3**) may be used to obtain the following:

Theorem *Consider the axisymmetric fluid domain W illustrated in Fig. 1 where the torus D is strictly bounded by two vertical cylinders that intersect D at the free surface F (the so called 'John' condition); the inner cylinder has radius b. For a given azimuthal mode number n, suppose that for some value of the non-dimensional frequency parameter Kb*

$$j_{n,m} \le Kb \le j'_{n,m+1}, \tag{6}$$

where $j_{n,m}$ denotes the m-th zero of the Bessel function J_n and $j'_{n,m}$ denotes the m-th zero of J'_n. If $n = 0$, then $m \in \{1,2,\dots\}$. If $n \ge 1$, then $m \in \{0,1,\dots\}$ with $j_{n,0} = 0$. Then, for this value of Kb, the boundary-value problem (1-4) has only trivial solutions in the form (5).

In other words, for toroidal geometries satisfying the John condition, the solution of the water wave problem with azimuthal mode number n is unique provided that (6) is satisfied.

4 Trapped mode solutions

Trapped mode solutions to the problem (1-4) are sought in the form (5) where $\phi^{(n)}$ is taken as the potential of a ring source of radius c in the free surface. The potential for such a source (**8**, equation 3.10) singular on $(r,y) = (c,0)$ is

$$\begin{aligned} R_n(r,y;c) &= 4\pi^2 iKc\,\mathrm{e}^{-Ky} J_n(Kr_<)H_n^{(1)}(Kr_>) \\ &\quad + 8c\int_0^\infty (\nu\cos\nu y - K\sin\nu y) I_n(\nu r_<)K_n(\nu r_>)\frac{\nu\,\mathrm{d}\nu}{\nu^2+K^2}, \end{aligned} \tag{7}$$

where $r_> = \max\{r, c\}$, $r_< = \min\{r, c\}$, and J_n, I_n, K_n and $H_n^{(1)}$ denote standard Bessel, modified Bessel and Hankel functions of order n. In general, at large radial distances the ring source gives outgoing waves as a result of the Hankel function in the first term. It may be shown that the integral term decays like r^{-3} as $r \to \infty$. Radiating waves in $r > c$ are annulled by taking c to satisfy $J_n(Kc) = 0$; that is Kc is chosen to be a zero of the Bessel function J_n. Any surface in the fluid domain that is always parallel to the local velocity may be interpreted as the surface of a structure.

The purely axisymmetric case $n = 0$ was considered by McIver & McIver (**7**). It was proved using the Stokes' stream function that for a given non-radiating ring source a family of corresponding toroidal structures can be constructed that exclude the source from the fluid domain, thus establishing the existence of trapped mode solutions. For $n \geq 1$ no stream function is available so here the evidence given for the existence of trapped modes is purely numerical.

On the surface of any structure it is required that there is no flow in the normal direction. For axisymmetric structures, surfaces independent of θ are sought in the form $r = r(y)$ and the condition of no flow in the direction of the local normal $\mathbf{n}$ may be written

$$\frac{\mathrm{d}r}{\mathrm{d}y} = \frac{\phi_r}{\phi_y}. \tag{8}$$

This differential equation may be solved numerically using standard procedures. A typical calculation is given in Fig. 2 for the case of azimuthal mode number $n = 1$, the radius of the ring source is chosen as its smallest possible value $Kc = j_{1,1}$. The figure shows typical stream surfaces; any surface, or combination of surfaces, for which the singularity is enclosed may be interpreted as a structural surface. It should be noted that these stream surfaces do not satisfy the John condition required by the theory of §3.

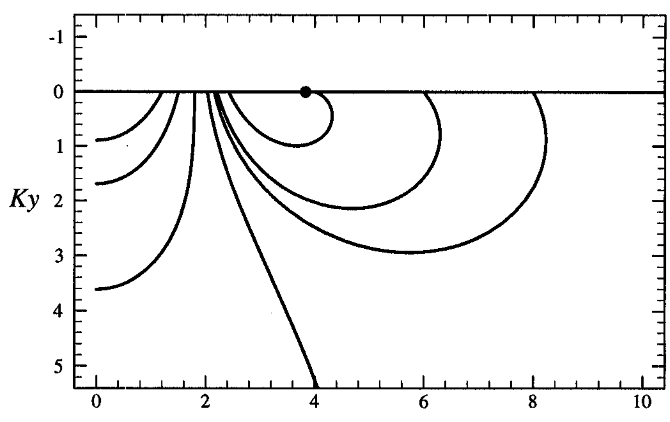

Figure 2: Axial plane cross section of stream surfaces for mode number $n = 1$. The source position $Kc = j_{1,1}$ is marked •.

5 Discussion

Fig. 3 shows numerical calculations of the values of Kb corresponding to the intervals of existence for trapped modes that may be constructed using a single ring source. Also shown are the intervals for uniqueness given in (6). These intervals are complementary despite the fact that the trapped mode solutions violate the conditions under which the uniqueness theorem was derived.

It is apparent from that there are intervals in Kb for which there may be uniqueness for all modes (disregarding for the moment the requirement that John condition must be satisfied). For example, for $n = 1$ no trapped modes have been found for $Kb \in (2.51, 3.05)$ while uniqueness of the

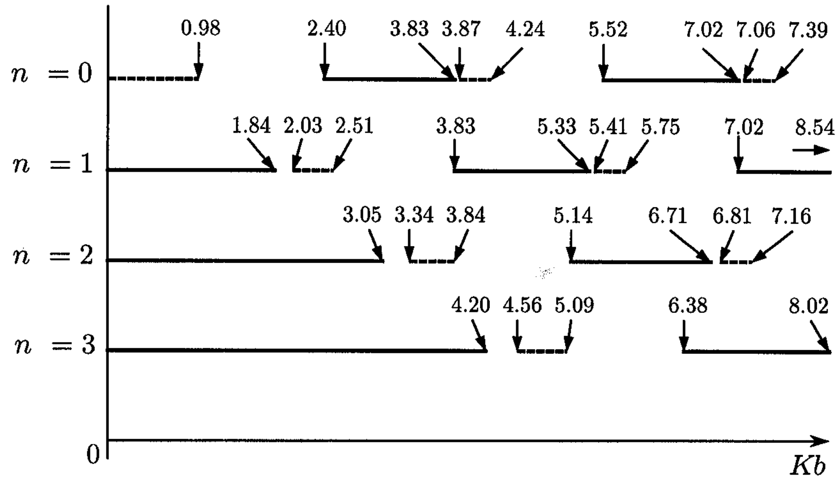

Figure 3: Values of the inner radius Kb for which uniqueness has been established (————), provided the structure satisfies the John condition, and values of Kb for which trapped modes may be constructed (– – – – –) using a single ring source. The integer n is the azimuthal wave number.

solution has been established in this interval for all other modes. However, it should be pointed out that in the equivalent two-dimensional problem of two-surface piercing bodies, Linton & Kuznetsov (**9**) have found evidence of modes trapped by bodies violating the John condition within the region for which uniqueness is predicted by the theory that requires the John condition.

6 Conclusion

The uniqueness of the solution to linear water-wave problems with axisymmetric floating bodies has been considered. Uniqueness of the solution has been established for a restricted class of body geometries over certain ranges of frequency. Further, examples of non-uniqueness, or trapped modes, have been constructed numerically for geometries that do not satisfy the restrictions required by the uniqueness theorem but, nevertheless, the frequency ranges where they occur are entirely consistent with that theorem. Further work is required to extend the uniqueness theorem to a wider class of geometries and to explore the range of trapped mode solutions that are possible.

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DISCUSSION

Clark P.: When engineers hear about uniqueness and trapped modes they instinctively think that these are issues of concern to mathematicians only. Could the axisymmetric wave trap described in your paper be the basis of an effective wave power device, where the energy in the trapped modes would be extracted by, say, a Well's turbine? This woul be of great interest to engineers.

McIver P., Kuznetsov N.: Resonance will occur when forcing is applied at a trapped mode frequency. Whether the bandwidth around a trapped mode frequency is substantially different from that for the near-resonant motions already familiar to designers of wave-power devices is still an open question, but one that we will investigate in the near future. There is certainly a possibility that improved oscillating water column devices may result from this work.
Another reason the existence of pure (as opposed to leaky) trapped modes should be of concern to engineers is because standard numerical methods will fail at, or very close to, a trapped mode frequency.

Eatock Taylor R.: Is it significant that these bulbous shapes are not "wall-sided" at the water line? Is the trapped modes "industry" interested in wall-sided bodies which are concave below the water-line? Such structures can presumably experience cancellation in the vertical wave force.

McIver P., Kuznetsov N.: In the water-wave problem, there is still a great deal to be understood about the circumstances under which trapped modes can occur. In particular, it is not known whether the first examples of "open-sea" trapped modes presented at this workshop have geometries which are in some way typical; their particular character may be just a result of the method of construction. We certainly cannot rule out the existence of trapped modes for the type of wall-sided bodies you describe.