Recent results on trapped modes and their influence on finite arrays of vertical cylinders in waves

R. Porter and D. V. Evans
School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK

Introduction

At the last Workshop in Hamburg, Maniar & Newman showed that at particular frequencies the in-line first-order exciting forces on those cylinders near the centre of a large number of identical bottom-mounted vertical circular cylinders in a linear array become extremely large, compared to the force on an isolated cylinder. These frequencies coincide with those associated with certain trapped modes around a corresponding cylinder on the centre-plane of a wave channel. These trapped modes are of two types. The Neumann trapped modes, satisfying Neumann conditions on all solid boundaries and a Dirichlet condition on the centre-plane, were discovered by Callan *et al* (1991) and have been proved to exist for all values of $0 < a/d < 1$ where $2a$ is the cylinder diameter and $2d$ is the width of the channel. See Evans *et al* (1994). Numerical computations by Callan *et al* (1991) indicate that there is just one such trapped mode having a unique wavenumber, k^N, satisfying $k^N < \pi/2d$ where the angular velocity ω^N is given by $\omega^N = (gk^N \tanh k^N h)^{1/2}$ with h the depth of the channel. Physically the Neumann trapped mode describes an antisymmetric sloshing motion about the centre-plane of the channel which is confined to the vicinity of the cylinder and decays rapidly down the channel. Mathematically, the value $(k^N)^2$ is an eigenvalue of the Laplacian operator in the unbounded region contained between one channel wall, the centre-plane of the channel and one half of the cylinder, and $(k^N)^2$ lies below the continuous spectrum which for this problem is $[\pi^2/4d^2, \infty)$.

The second type, discovered by Maniar & Newman (1996) and described as Dirichlet trapped modes, satisfy a Neumann condition of no normal flow through the cylinder surface but Dirichlet conditions on both the channel walls and the centre-plane. They have no obvious physical interpretation in the context of water waves but are well-known in the acoustical literature where they are termed acoustic resonances. For a review, see Parker & Stoneman (1989).

In contrast to the Neumann trapped modes, the Dirichlet trapped modes only appear to exist for a restricted range of a/d. Thus the computations of Maniar & Newman (1996) suggest that a Dirichlet trapped mode exists provided $0 < a/d \lesssim 0.677$, a figure which the present authors have refined to 0.6788 using the same method.

Convincing experimental evidence for the Neumann trapped modes have been given recently by Retzler (private communication, 1996). Maniar & Newman (1996) pointed out that long finite periodic arrays of identical bottom-mounted cylinders have applications to structures such as long bridges or proposed designs for offshore airports. In practice, however, it is clear that at least a double array of supporting cylinders will be needed so that it is important to predict the corresponding trapped mode frequencies for *more* than a single cylinder on the centre-line. In fact we have solved the problem of determining all the trapped modes which can occur when any number of rigid bottom-mounted vertical circular cylinders are placed on the centre plane of a channel. The cylinders can have any radii and can be spaced arbitrarily and the trapped modes are antisymmetric about the centre plane and satisfy either Neumann or Dirichlet conditions on the channel walls.

The method is based on the multipole method, in which singular solutions of the Helmholtz equation satisfying an antisymmetry condition on the channel centre plane are modified to include the boundary conditions on the channel walls. The total potential about any cylinder may then be expressed as a Fourier-type sum over all relevant multipoles and the total potential anywhere in the channel as the sum over all cylinders. The remaining condition to be satisfied, that of no-flow on

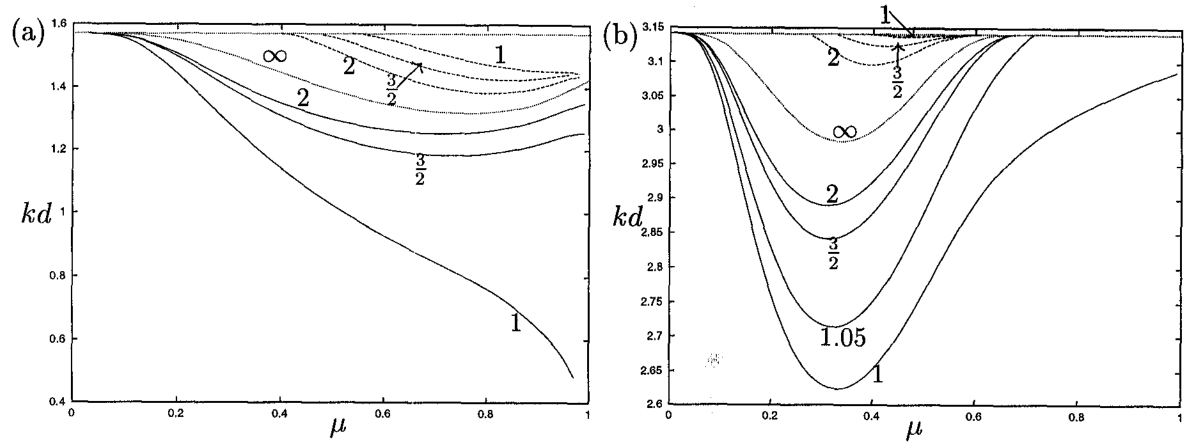

Figure 1: Variation of (a) Neumann and (b) Dirichlet trapped mode frequencies as μ varies in the case of two cylinders for different values of the spacing parameter λ (shown against the curves). Symmetric modes (——), antisymmetric modes (– – –).

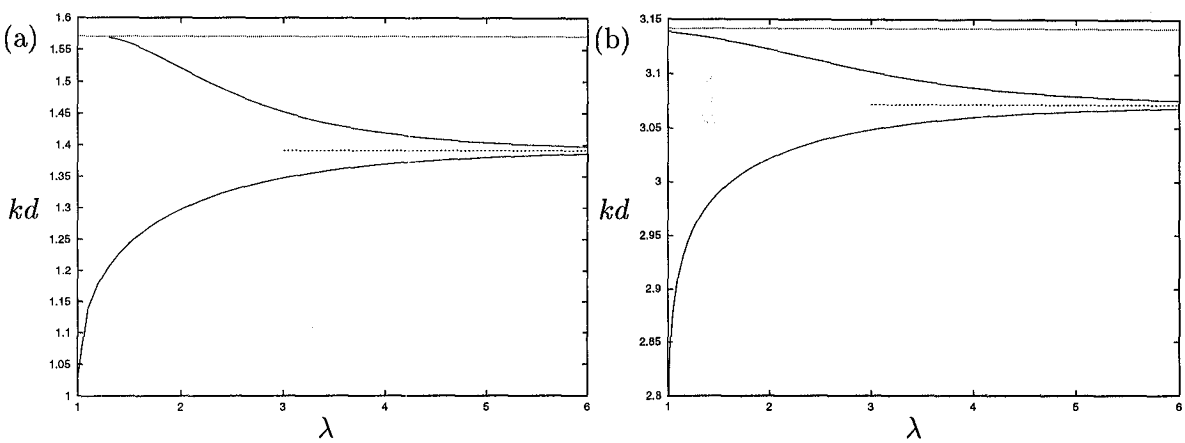

Figure 2: Variation of (a) Neumann and (b) Dirichlet trapped mode frequencies for two cylinders, both $\mu = \frac{1}{2}$ as the spacing parameter, λ, varies.

the cylinder surfaces, is achieved by use of a Bessel function addition theorem, as in Linton & Evans (1990), yielding a homogeneous determinant system whose non-trivial solutions correspond to the trapped mode frequencies. The same method has recently been used by Linton & McIver (1996) to determine the scattering properties of any number of circular cylinders of arbitrary size and position in a channel. The method is an extension of that used by Callan *et al* (1991) for the single cylinder.

The number of possible configurations of cylinders we could consider to illustrate the results is of course limitless so we shall concentrate mainly on the case of two identical cylinders because of its connection with *finite* double arrays of cylinders which occur in offshore structures. The non-dimensional trapped mode wavenumber kd is in this case a function of two dimensionless parameters μ and λ describing the size and spacing of the centres of the cylinders. We choose $\mu = a/d$ and let the centres of the cylinders be located at $(\pm\lambda a, 0)$ so that λ is a spacing parameter being the ratio of cylinder separation to cylinder diameter. When $\lambda = 1$ the cylinders are touching and as $\lambda \to \infty$ we would expect results for the trapped modes to approach the single cylinder results as the interaction between them diminishes. This proves to be the case as figure 1 illustrates. Here, Neumann and Dirichlet trapped mode wavenumbers kd are plotted against μ for different λ. Also shown is the unique curve for both the Neumann and Dirichlet trapped modes for an isolated cylinder which we label $\lambda = \infty$. We consider the Neumann modes first, all of whose wavenumbers satisfy $kd < \pi/2$. The solid curves are symmetric Neumann trapped modes whilst the dashed curves above the $\lambda = \infty$ curve are all antisymmetric Neumann trapped modes. We can draw the following conclusions about the Neumann modes from figure 1. For sufficiently large μ there exist *two* trapped modes, a low frequency symmetric mode and a higher frequency antisymmetric mode, for each value of λ. However, for fixed

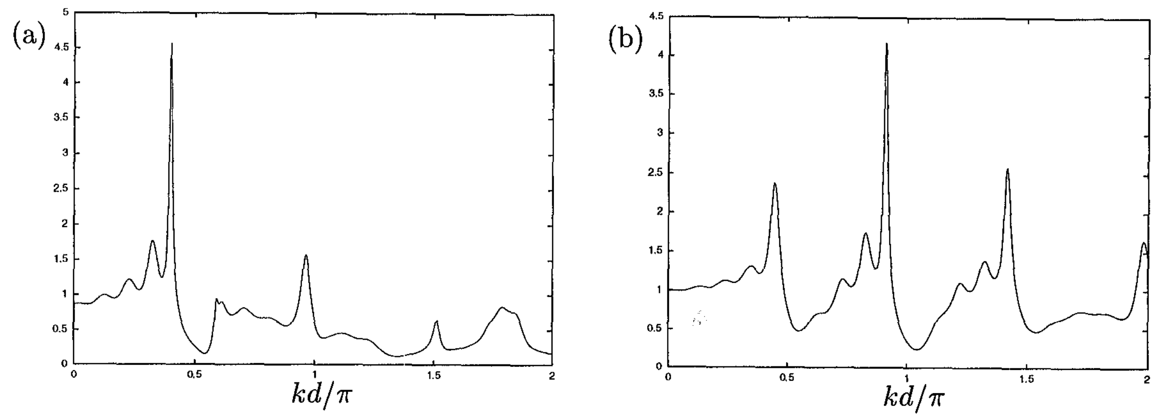

Figure 3: Maximum exciting force against non-dimensional wavenumber kd in the case of head seas interacting with a double array of 2×9 cylinders all of radius a. The two rows are $4a$ apart and in each row the centres are $2d$ apart. (a) $a/d = \frac{1}{2}$, (b) $a/d = \frac{1}{4}$.

λ, as μ decreases a value is reached at which the antisymmetric mode disappears.

It appears from figure 1 that, as expected, the curves for increasing λ approach the single cylinder results. This is more clearly seen in figure 2 which plots kd against λ for $\mu = \frac{1}{2}$. Both the Dirichlet and the Neumann curves rapidly approach the corresponding single cylinder trapped mode frequency as λ increases. Notice how the antisymmetric Neumann mode cuts off below a certain value of $\lambda > 1$ whilst the antisymmetric Dirichlet mode persists down to touching at $\lambda = 1$. This behaviour can also be seen from figure 1 by considering the intersection of $\mu = \frac{1}{2}$ with curves of different λ. However it is also clear from figure 1 that in general the behaviour of the Dirichlet modes is more complicated than the Neumann modes.

It is possible to remove the channel walls and regard both types of trapped modes as oscillations between adjacent pairs of cylinders in a doubly-infinite row, the Neumann modes having an antinode at each mid-plane between pairs of cylinders and the Dirichlet modes a node. Following the discussion of Maniar & Newman (1996) that a *finite* single row containing many cylinders could experience large forces and at frequencies close to the Neumann and Dirichlet trapped modes for a single cylinder in a channel, or its equivalent *infinite* row of cylinders, we should expect that the peaks in figures 3(a) and 3(b) which give the maximum in-line exciting force on the middle pair of cylinders in a double row of 2×9 cylinders due to head seas to be close to the corresponding symmetric trapped modes. In figures 3(a),(b) the distance between two cylinders in a pair is $4a$ so that in both figures the corresponding doubly-infinite row requires $\lambda = 2$. It is clear from figure 1 at $\lambda = 2$ that this is indeed the case. Thus the computed values of the symmetric Neumann and Dirichlet trapped mode wavenumbers for $\mu = \frac{1}{2}$ are $kd = 1.29771$ and $kd = 3.02157$ respectively compared to the peaks at 1.256 and 3.024 in figure 3(a) whilst for $\mu = \frac{1}{4}$ the trapped modes at $kd = 1.46567$ and 2.90894 compare to the peaks at 1.400 and 2.856 respectively in figure 3(b). The other peaks in figures 3(a),(b) correspond to nearly-trapped waves. However, recent careful numerical calculations have confirmed, using two independent methods that it is possible to find a pure trapped mode of both Neumann and Dirichlet type, at the very precise values of $a/d = 0.3520905$ with $k^N d = 1.488884\pi$ and for $a/d = 0.2670474$ with $k^D d = 1.991867\pi$ respectively. Notice that these trapped modes are embedded in the continuous wavenumber spectrum. Surface elevations for each trapped mode are presented in figure 4.

The effect of increasing the number of cylinders in a channel is generally to increase the number of trapped mode frequencies to be equal to the number of cylinders. It can be anticipated that the exciting force on cylinders in the centre of an array consisting of four lines of periodically-spaced identical cylinders would show large values at the four wavenumbers corresponding to the equivalent trapped modes.

Finally it is not necessary to have large numbers of cylinders to obtain large forces. As a result of near-trapping effects it is possible for as few as four bottom-mounted cylinders to manifest large forces, in this case, 50 times that on a single cylinder, when excited by a particular incident wave

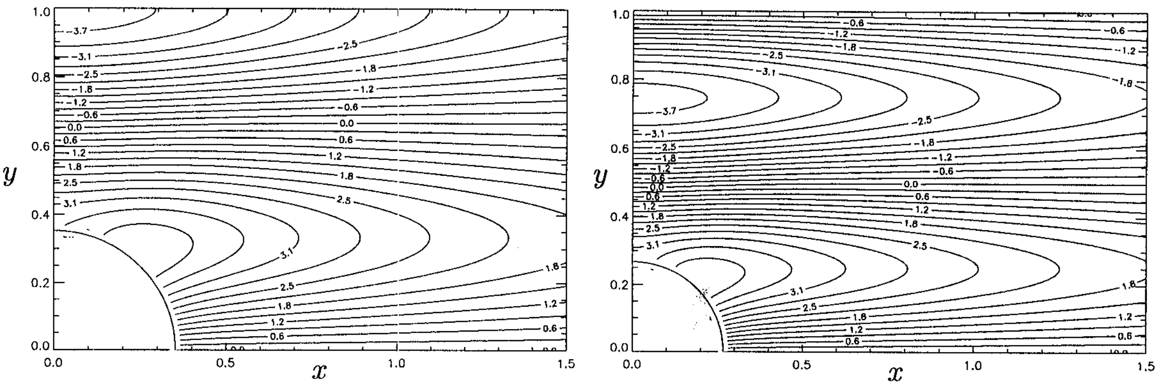

Figure 4: Surface elevation for Neumann and Dirichlet trapped modes embedded in the continuous spectrum.

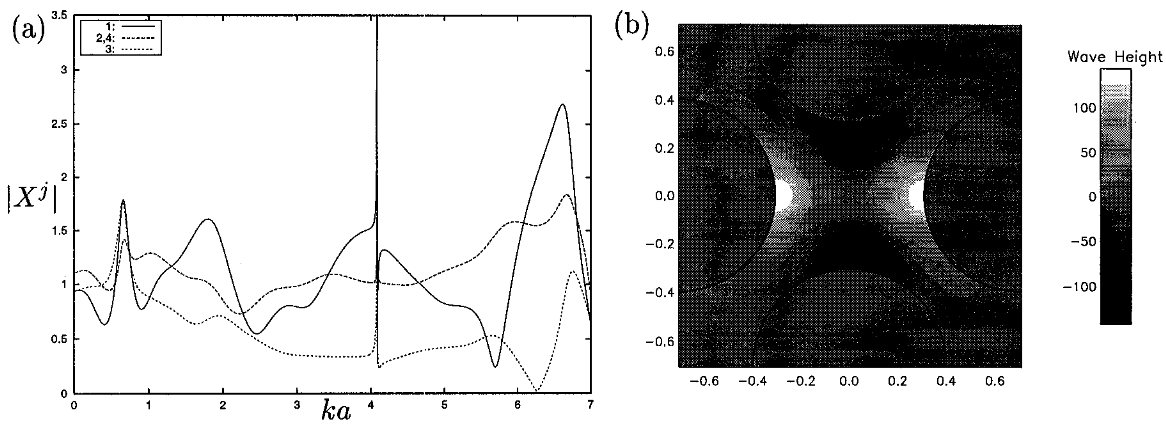

Figure 5: (a) Non-dimensional forces on four cylinders arranged in a square in head seas as wavenumber ka varies; (b) Free surface elevation at near-trapping ($ka = 4.08482$).

frequency, as shown in figure 5(a) using the interaction theory of Linton & Evans (1990). Notice from figure 5(b), which describes the free surface at near-trapping that the force acts radially and alternates in sign from one cylinder to the next.

Conclusion

We have illustrated the importance of an understanding of trapped modes in order to anticipate possible large forces on finite arrays of bottom-mounted cylinders. We have derived results for any number of cylinders in a wave channel and shown how these affect a finite double array of cylinders. We have also shown how near-trapping can account for large forces for as few as four cylinders when the spacing is sufficiently small, and we have presented results for pure trapped modes which are embedded in the continuous spectrum. Further results will be described at the Workshop.

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