Super Green Functions for Generic Dispersive Waves

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Green functions and super Green functions

In potential flows, a Green function $G(\vec{\xi},\vec{x})$ defines the velocity potential of the flow created at a point $\vec{\xi}=(\xi,\eta,\zeta)$ by a source of unit strength located at a point $\vec{x}=(x,y,z)$. The Green function for an unbounded incompressible fluid is

$$4\pi G = -1/r$$

where $r=\sqrt{(\xi-x)^2+(y-\eta)^2+(z-\zeta)^2}$ is the distance between the field point $\vec{\xi}$ and the singular point $\vec{x}$. In free-surface hydrodynamics, Green functions can be expressed as

$$G = G^S + G^F$$

where G^F accounts for free-surface effects and G^S is defined in terms of simple singularities. E.g., for time-harmonic ship waves in deep water, the simple-singularity component G^S is given by

$$4\pi G^S = -1/r + 1/r'$$

where $r'=\sqrt{(\xi-x)^2+(y-\eta)^2+(z+\zeta)^2}$. The free-surface component G^F is given by the Fourier superposition of elementary waves

$$4\pi^2 G^F = \lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \, \frac{e^{Zk-i(X\alpha+Y\beta)}}{D+i\,\varepsilon\,\mathrm{sign}(D_f)} \qquad \text{(1a)}$$

where $k=\sqrt{\alpha^2+\beta^2}$ is the wavenumber and

$$(X,Y,Z\le 0) = (\xi-x,\eta-y,\zeta+z) \qquad \text{(1b)}$$

Furthermore, D is the dispersion function

$$D = (f-F\alpha)^2 - k$$

and $\mathrm{sign}(D_f)=\mathrm{sign}(\partial D/\partial f)$ is given by

$$\mathrm{sign}(D_f) = \mathrm{sign}(f-F\alpha)$$

The nondimensional frequency f and the Froude number F are defined as

$$f=\omega\sqrt{L/g} \qquad F=\mathcal{U}/\sqrt{gL}$$

where ω is the encounter frequency of the regular ambient waves exciting the ship motions, L and $\mathcal{U}$ are the ship length and forward speed, and g is the acceleration of gravity.

Two fundamental difficulties greatly restrict the practical utility of free-surface Green functions. A first major difficulty is that the singular double Fourier integral representation (1a) of free-surface effects is nearly impossible to compute accurately (except in very few relatively simple cases for which the near-field behavior of G^F can be determined analytically [1]) in the critically-important limit $(X,Y,Z)\to 0$ where (1a) has a very complex singularity. A second major difficulty is that although Green functions provide valuable insights, they are not directly useful (except for idealized cases involving flows about a sphere) for practical applications, which require flows generated by *distributions* of singularities (sources and dipoles) over hull-panels and waterline-segments. Indeed, practical calculations involve *distributions* of singularities (rather than point singularities) of the form

$$\mathcal{G} = \int_{P_0}\left\{\begin{matrix} G\sigma \\ \nabla G\cdot\vec{\delta}\end{matrix}\right\} \qquad (2)$$

where P_0 stands for a hull-panel or a waterline-segment near a point[1] $\vec{x}_0=(x_0,y_0,z_0\le 0)$, and σ and $\vec{\delta}=(\delta_x,\delta_y,\delta_z)$ are source and dipole densities, respectively. A function (2) associated with a *distribution* of singularities is called a *super Green function* to emphasize its similarities and differences with usual Green functions associated with *point* singularities. Evaluation of super Green functions $\mathcal{G}$ for free-surface flows in the usual approach, in which G and ∇G are evaluated using (1) and subsequently integrated over a panel or a segment as in (2), is a hopeless task which cannot be performed accurately (notably for time-harmonic ship waves) for field points near a waterline-segment or a hull-panel at the free surface.

Fourier-Kochin representation of super Green functions

However, the super Green functions $\mathcal{G}$ of main interest in free-surface hydrodynamics, and their first[2] derivatives $\nabla\mathcal{G}$, can be evaluated in a remarkably simple way using Kochin's formulation and the Fourier representation of free-surface effects summarized below. Within the Fourier-Kochin formulation [2], the free-surface-effect component

$$\mathcal{G}^F = \int_{P_0}\left\{\begin{matrix} G^F\sigma \\ \nabla G^F\cdot\vec{\delta}\end{matrix}\right\} \qquad (3)$$

[1]The reference point $\vec{x}_0$ may be taken at (or near) the centroid of P_0

[2]and indeed higher

of the super Green function

$$\mathcal{G} = \mathcal{G}^S + \mathcal{G}^F$$

is defined by substituting (1) into (3) and performing the space integration over the hull-panel or the waterline-segment before the Fourier integration. Thus, the free-surface component $\mathcal{G}^F$ is given by the double Fourier integral representation

$$4\pi^2\mathcal{G}^F = \lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \, \frac{S\, e^{Zk\,-\,i\,(X\alpha+Y\beta)}}{D+i\,\varepsilon\,\mathrm{sign}(D_f)} \quad \text{(4a)}$$

where

$$(X\,,Y,Z\leq 0) = (\xi - x_0\,,\eta - y_0\,,\zeta + z_0) \quad \text{(4b)}$$

and S is the spectrum function defined as

$$S = \int_{\mathrm{P}_0} \mathcal{E} \left\{ \frac{\sigma}{i\,\alpha\,\delta_x + i\,\beta\,\delta_y + k\,\delta_z} \right\} \quad \text{(5a)}$$

$$\text{with} \quad \mathcal{E} = e^{k\,(z-z_0)\,+\,i\,[\,\alpha\,(x-x_0)+\beta\,(y-y_0)\,]} \quad \text{(5b)}$$

The integral representations (1) and (4) of the free-surface components G^F and $\mathcal{G}^F$ of the related Green function G and super Green function $\mathcal{G}$ show that G^F is a special case of $\mathcal{G}^F$ corresponding to

$$S = 1$$

An essential property of the spectrum function (5) associated with a *distribution* of singularities is

$$S \to 0 \quad \text{as} \quad k = \sqrt{\alpha^2 + \beta^2} \to \infty$$

As a result, the super Green function $\mathcal{G}^F$ defined by (4) is not singular in the limit $(X,Y,Z) \to 0$, unlike G^F which has a complex singularity in this limit. Furthermore, space integration over a hull-panel or a waterline-segment is incomparably simpler in (5a), where the elementary wave-function (5b) is infinitely differentiable, than in (3) which involves functions G^F and ∇G^F singular in the limit $(X,Y,Z) \to 0$. Thus, the Fourier-Kochin representation of super Green functions given by (4) and (5) effectively circumvents the two previously-noted fundamental difficulties restricting the utility of the classical approach based on (1) and (3). In this usual approach, *influence coefficients*[3] in fact cannot be evaluated with accuracy for field (control) points in the vicinity of a distribution of singularities over a waterline segment or a hull-panel at the free surface. However, the Fourier-Kochin representation (4) and (5) makes it possible to evaluate influence coefficients in all cases, including the most difficult and important[4] case involving a waterline segment or a hull-panel at the free surface.

[3] which are super Green functions

The space integration (5) in the Fourier-Kochin representation of super Green functions is a trivial task, as was already noted. However, the Fourier integration (4a) is a nontrivial issue. This critical issue is considered in [3-5] and in a forthcoming study [6] for an arbitrary spectrum function S and an arbitrary dispersion function D, i.e. for generic dispersive waves generated by an arbitrary distribution of singularities. Indeed, while super Green functions are defined above for time-harmonic ship waves in deep water, a broad class of dispersive waves, including steady or time-harmonic water waves in finite water depth (with or without forward speed) and internal waves in a density-stratified fluid, are defined by the generic Fourier representation (4). The most important results given in [3] and [5] and in the unpublished study [6] are summarized here.

Far-field waves

The generic super Green function $\mathcal{G}^F(X,Y)$ defined by the Fourier representation

$$\mathcal{G}^F = \lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \, \frac{A\, e^{-i\,(X\alpha+Y\beta)}}{D+i\,\varepsilon\,\mathrm{sign}(D_f)} \quad \text{(6)}$$

is now considered for generic dispersion and amplitude functions D and A. We may assume that the amplitude function $A(\alpha,\beta)$ in (6) vanishes as $k \to \infty$ and does not oscillate rapidly, as follows from (5). We have

$$\mathcal{G}^F \sim \mathcal{G}^W \quad \text{as} \quad H = \sqrt{X^2+Y^2} \to \infty \quad \text{(7)}$$

where $\mathcal{G}^W$ represents the far-field waves contained in $\mathcal{G}^F$. The far-field waves $\mathcal{G}^W$ are given by single Fourier integrals along curves, called *dispersion curves*, defined by the dispersion relation[5] $D=0$:

$$\mathcal{G}^W = -i\pi \sum_{D=0} \int_{D=0} ds \, \frac{\Theta\, A}{\|\nabla D\|}\, e^{-i\,(X\alpha+Y\beta)} \quad \text{(8a)}$$

$$\text{with} \quad \Theta = \mathrm{sign}(D_f) + \mathrm{erf}\left(\frac{k_*(XD_\alpha + YD_\beta)}{\sigma\,\|\nabla D\|}\right) \quad \text{(8b)}$$

Here, $\sum_{D=0}$ means summation over all the dispersion curves, ds is the differential element of arc length of a dispersion curve, $\|\nabla D\|^2 = D_\alpha^2 + D_\beta^2$ with $D_\alpha = \partial D/\partial\alpha$ and $D_\beta = \partial D/\partial\beta$, erf is the

[4] because free-surface effects are largest in this case

[5] The dispersion relation typically defines several distinct dispersion curves, although a single dispersion curve may exist in simple cases; e.g. wave diffraction-radiation without forward speed

usual error function, σ is a positive real constant whose role is explained further on, and k_* is a reference wavenumber. The reference wavenumber k_* may be taken equal to the local value of the wavenumber k at the dispersion curve, although other reference wavenumbers may be used. E.g., for free-surface flows, $k_* = f^2$ and $k_* = 1/F^2$ are proper choices for time-harmonic flows without forward speed and steady flows, respectively.

In the far-field limit $H \to \infty$, (8) yields

$$\mathcal{G}^W \sim -i\pi \sum_{D=0} \int_{D=0} ds\,[\text{sign}(D_f) + \text{sign}(\vec{X}\cdot\nabla D)]\; A\exp[-i\,(X\alpha + Y\beta)]/\|\nabla D\| \qquad (9)$$

Expressions (9) and (8), given in [3] and [5] respectively, are asymptotically equivalent Fourier representations of the far-field waves $\mathcal{G}^W$ contained in $\mathcal{G}^F$. The radiation condition is satisfied via the sign function $\text{sign}(D_f)$, which stems from the $\varepsilon \to +0$ limit in (6). Expression (9) is independent of the constant σ in (8). The far-field Fourier representation (9) is applied to the important case of time-harmonic ship waves in deep water in [7]. This Fourier integral representation of far-field waves in generic dispersive media can be further approximated using the method of stationary phase. The stationary-phase approximation of (9) yields simple relations, given in [8], between the dispersion curves associated with the dispersion relation $D=0$ and important aspects (wavelengths, directions of wave propagation, phase and group velocities, and cusp angles) of the corresponding far-field waves.

Wave and local components

In the near field, the super Green function (6) can be expressed as the sum of a wave component $\mathcal{G}^W$ and a local (near-field) component $\mathcal{G}^N$:

$$\mathcal{G}^F = \mathcal{G}^W + \mathcal{G}^N \qquad (10)$$

where $\mathcal{G}^W$ is given by (8). The positive real constant σ in (8) may be chosen so that the local component $\mathcal{G}^N$ decays without oscillations, i.e. so that the wave component $\mathcal{G}^W$ fully accounts for the waves included in $\mathcal{G}^F$ in the *near* field (as well as in the far field where $\mathcal{G}^N$ is negligible and $\mathcal{G}^F \sim \mathcal{G}^W$). Thus, both the wave component $\mathcal{G}^W$ and the local component $\mathcal{G}^N$ in (10) involve the constant σ, although the sum $\mathcal{G}^W + \mathcal{G}^N$ is of course independent of σ, like the representation (9) of the far-field waves contained in $\mathcal{G}^F$.

The decomposition (10) into wave and local components is nonunique. The wave component $\mathcal{G}^W$ in (10) is taken as the representation (9) in [3] and [4], where a Fourier representation of the corresponding local component $\mathcal{G}^N$ suited for numerical evaluation is given. In the present study, the wave component $\mathcal{G}^W$ in (10) is taken as the representation (8) obtained in [5]. The wave component (9) used in [3] and [4] is a particular case[6] of the wave component (8). The integrand of the double Fourier integral representation of the local component $\mathcal{G}^N$ given in [4] is continuous everywhere but varies rapidly across a dispersion curve. Here, the more general expression (8) for the wave component is used, and a remarkably simple Fourier representation of the corresponding local component $\mathcal{G}^N$ is given. The near-field representation of $\mathcal{G}^F$ given here is mathematically exact[7] and is quite well suited for accurate and efficient numerical evaluation. In particular, the integrand of the double Fourier integral representation of the local component $\mathcal{G}^N$ given further on is continuous everywhere and varies *slowly* across a dispersion curve.

Local component

Practical Fourier representations, suited for accurate and efficient numerical evaluation, of the wave component $\mathcal{G}^N$ associated with the Fourier representation of generic dispersive waves defined by (6), (10), (8) are given in [6] for generic dispersive waves and for the specific case of time-harmonic ship waves. A dispersion relation $D=0$ may define several basic types of dispersion curves, including *closed* dispersion curves surrounding points (α_j, β_j) in the Fourier plane and *open* dispersion curves. These various cases are considered in [6]. The case of a dispersion relation which yields open dispersion curves defined by single-valued functions $\alpha = \alpha_j(\beta)$ with $-\infty < \beta < \infty$ is considered here. Steady ship waves and wave diffraction-radiation by a ship for $\tau = \mathcal{U}\omega/g > \sqrt{2/27} \approx 0.272$, are examples of this type of dispersion curves, called open dispersion curves of type A. In this case[8] the wave and local components in (10) are given by

$$\mathcal{G}^W = -i\pi \sum_j \int_{-\infty}^{\infty} d\beta\,[\text{sign}(D_f D_\alpha) + \text{erf}(\frac{k_* X}{\sigma})]\; A\exp[-i\,(X\alpha + Y\beta)]/D_\alpha$$

Here σ and k_* are the positive real constant and the reference number already introduced in (8b). The

[6]Expression (9) corresponds to the far-field limit $H \to \infty$ or the limit $\sigma \to 0$ of (8)

[7]whereas the representation of $\mathcal{G}^N$ given in [4] involves numerical approximations

[8]for which constant-β lines intersect each dispersion curve only once

relation $d\beta/|D_\alpha| = ds/\|\nabla D\|$ yields the alternative expression

$$\mathcal{G}^W = -i\pi \sum_j \int_{D=0} ds\,[\text{sign}(D_f) + \text{sign}(D_\alpha)\,\text{erf}(\frac{k_* X}{\sigma})] \\ A \exp[-i\,(X\alpha + Y\beta)]/\|\nabla D\|$$

The local component $\mathcal{G}^N$ is given by

$$\mathcal{G}^N = \int_{-\infty}^{\infty} d\beta\, e^{-iY\beta} \int_{-\infty}^{\infty} d\alpha\, e^{-iX\alpha} (\frac{A}{D} - \sum_j \frac{E_j^\alpha A_j}{(\alpha-\alpha_j) D_\alpha^j})$$

where A_j and D_α^j stand for the values of the functions A and D_α at the j^{th} dispersion curve[9] $\alpha = \alpha_j(\beta)$, and E_j^α is the localizing function

$$E_j^\alpha = \exp[-\frac{\sigma^2}{4}(\frac{\alpha-\alpha_j}{k_j^*})^2]$$

Here k_j^* is the reference wavenumber attached to the j^{th} dispersion curve. The integrand of the double Fourier integral representation of the local component $\mathcal{G}^N$ is finite at a dispersion curve. Specifically, we have

$$\frac{A}{D} - \frac{E_j^\alpha A_j}{(\alpha-\alpha_j) D_\alpha^j} \sim \frac{A_\alpha^j D_\alpha^j - A_j D_{\alpha\alpha}^j/2}{(D_\alpha^j)^2} \quad \text{as } \alpha \to \alpha_j \,.$$

where A_α^j and $D_{\alpha\alpha}^j$ are the values of A_α and $D_{\alpha\alpha}$ at the j^{th} dispersion curve. Furthermore, the localizing function E_j^α, and consequently the integrand of the Fourier representation of $\mathcal{G}^N$, vary slowly across a dispersion curve because it is not necessary to use small values[10] of the constant σ.

As was already noted, the foregoing Fourier representation of the super Green function (6) may be used for open dispersion curves of type A. An analogous Fourier representation may be used if the dispersion function D yields one or more dispersion curve, called open dispersion curves of type B, defined by single-valued functions $\beta = \beta_j(\alpha)$ with $-\infty < \alpha < \infty$. This Fourier representation of the super Green function (6) is given in [6], where similar expressions for the case of a closed dispersion curve and dispersion curves of arbitrary shape are also given. Applications of these Fourier representations of generic dispersive waves to the case of time-harmonic free-surface flows with forward speed may also be found in [6].

[9] E.g., for steady ship waves and wave diffraction-radiation by a ship for $\tau > \sqrt{2/27} \approx 0.27$ we have two distinct dispersion curves, and therefore $j = 1, 2$

[10] whereas the representation given in [4] requires thin dispersion strips, corresponding to very small values of σ

Conclusion

The foregoing Fourier representation of the generic super Green function (6) is remarkable in view of its generality[11], its simplicity and elegance, and the fact that it is well suited for accurate and efficient numerical evaluation[12]. Also, the decomposition (10) into wave and local components yields a natural decomposition of hydrodynamic loads into added-mass and wave-damping components in which damping effects are defined by single Fourier integrals.

Thus, free-surface Green functions, which offer important built-in advantages[13], can be used as effectively as simple Rankine sources. E.g., free-surface Green functions can be used in a calculation method based on linearization about the double-body flow[14], and to couple a nonlinear and/or viscous near-field calculation method and a far-field potential-flow representation [9].

References

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[11] The representation can be applied to a broad class of dispersive waves, including steady or time-harmonic water waves in finite water depth with forward speed and internal waves in a density-stratified fluid, generated by arbitrary distributions of singularities

[12] The integrands of the Fourier representations of both the wave component and the local component are continuous

[13] proper far-field behavior, radiation condition

[14] by distributing free-surface Green functions over the free surface in a finite, fairly small, region the *vicinity* of the ship since the Kelvin free-surface condition becomes nearly exact at a small distance from the ship