Trapped modes in wave channel with an elastic plate on the bottom

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1. Introduction

This present paper is concerned with trapped modes in a finite depth channel occupied by an inviscid, incompressible fluid under gravity with an elastic thin plate on the bottom. The mathematical model of the problem is considered in the framework of linearized water-wave theory.

The existence of trapped modes was first demonstrated by Ursell [1] in the case of an infinitely long, totally submerged cylinder in the infinite depth fluid. Further research in the trapped modes theory mainly concentrated on the investigation in the fluid either having a specific geometry of the bottom or including rigid bodies. The review of the more recent developments in the theory of trapped modes in water wave has been presented by Evans & Kuznetsov [2].

It should be noted that there exists another class of problems where the trapped modes phenomenon can also occur. It is the oscillation of the elastic body with inclusion having at least one infinite boundary. The possibility of trapped modes in the elastic systems was demonstrated in [4], [5], [6].

The similarity between these classes of the problems permits the possibility of the trapped modes phenomenon in "fluid-elastic body" systems. We consider such types of the bottom geometry for which no trapped modes exist, if the bottom is rigid [3]. The aim of the paper is to demonstrate that the elastic inclusion in the bottom can lead to the existence of trapped modes.

2. Statement of the problem

Consider the three-dimensional channel with the rectangular trench on the bottom. The bottom of the trench can be modeled by an infinite elastic thin plate. Cartesian axes are chosen so that y is directed vertically upwards and x and z are in the plane of the unperturbed bottom. The geometry is sketched in Fig.1. The motion of the fluid is described by velocity potential $\Phi(x,y,z,t)$ which must satisfy the boundary problem:

$$\nabla^2\Phi \quad \text{in the domain occupied by fluid,} \tag{1}$$

$$\Phi_{tt} + g\Phi_y = 0 \quad \text{on the free surface } y = H, \tag{2}$$

$$\Phi_y = w_t \quad \text{on the moving part of boundary,} \tag{3}$$

$$\partial\Phi/\partial n = 0 \quad \text{on the rigid part,} \tag{4}$$

where H is the depth of the channel, g is the acceleration due to gravity , w is the small plate displacement determined by the equation

$$Dw_{zzzz} + kw + Mw_{tt} = \rho\int_{-a}^{a}(\Phi_t + gw)dx \tag{5}$$

on the moving part of the boundary. Here M is the elastic construction mass per unit length, D is the cylindrical rigidity, k is the elastic foundation rigidity, ρ is the liquid density. Trapped

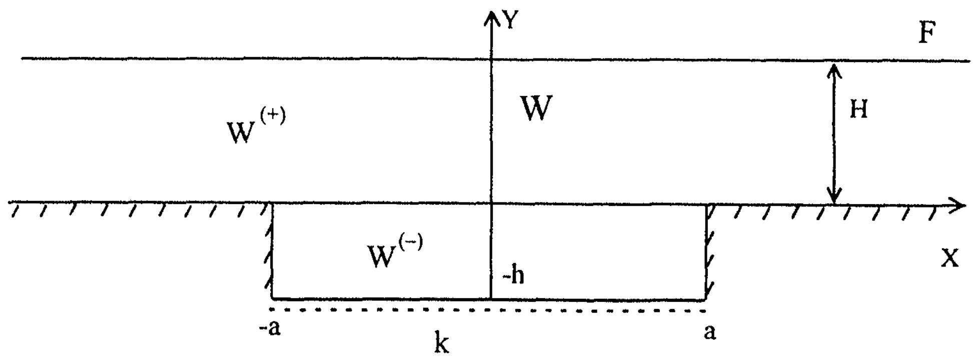

Figure 1: Sketch of geometry

mode solutions of the problem (1)-(5) are sought in the form

$$w(z,t) = \text{Re}\,\{w_0 e^{\mathrm{i}(mz-\omega t)}\}, \quad \Phi(x,y,z,t) = \text{Re}\,\{\varphi(x,y)e^{\mathrm{i}(mz-\omega t)}\}, \tag{6}$$

where ω is the frequency, m is the wavenumber, w_0 is an arbitrary constant. Substituting (6) into (1)-(5), that the function φ defined on a two-dimensional domain W which is a cross-section of fluid orthogonal to the z-axis, satisfies the following boundary value problem:

$$\nabla^2\varphi = m^2\varphi \ \text{ in } \ W, \quad \varphi_y = \frac{\omega^2}{g}\varphi \ \text{ when } \ -\infty < x < +\infty, \ y = H, \tag{7}$$

$$\varphi_y = -\mathrm{i}\omega w_0 \ \text{ when } \ |x| < a, \ \ y = -h, \quad \varphi_y = 0 \ \text{ when } \ |x| > a, \ \ y = 0, \tag{8}$$

$$\varphi_x = 0 \ \text{ when } \ x = \pm a, \ \ -h < y < 0, \tag{9}$$

$$(\tilde{k}_m - M\omega^2)w_0 = -\mathrm{i}\rho\omega\int_{-a}^{a}\varphi(x,y)dx, \quad |x| < a, \quad y = 0, \tag{10}$$

where $k_m = Dm^4 + k$ is the generalized rigidity of the plate, $\tilde{k}_m = k_m - 2ag\rho$. The radiation condition requires that φ and $\nabla\varphi$ decay at infinity in such a way that the energy of trapped fluid motion is finite and therefore

$$\int_W |\nabla\varphi|^2 dxdy + \frac{\omega^2}{g}\int_F \varphi^2 dx < \infty. \tag{11}$$

3. The case of an even bottom

For the even bottom, $h = 0$, we define a solution of the problem (7)-(9) with the help of the Green function which describes the fluid motion of a source placed on the bottom of the channel. The integral representation of the Green function is given in paper [6] and the Green function can be present as a series of exponents

$$G(|x|,y,\omega) = \begin{cases} A_0(y,\omega)e^{-\zeta_0|x|} + \sum_{k=1}^{\infty} A_k(y,\omega)e^{-\zeta_k|x|}, & \omega < \omega_b \\ iA_0(y,\omega)e^{i\zeta_0|x|} + \sum_{k=1}^{\infty} A_k(y,\omega)e^{-\zeta_k|x|}, & \omega > \omega_b, \end{cases} \tag{12}$$

where $A_0 = -\dfrac{2\xi_0\cosh y\xi_0}{\zeta_0(\sinh 2h\xi_0 + 2h\xi_0)}$, $A_k = -\dfrac{2\xi_k\cos y\xi_0}{\zeta_k(\sin 2h\xi_k + 2h\xi_k)}$;
ξ_0 is the positive root of the equation below

$$g\xi\tanh h\xi = \omega^2, \quad \zeta_0 = \begin{cases} \sqrt{m^2 - \xi_0^2}, & \omega < \omega_b, \\ \sqrt{\xi_0^2 - m^2}, & \omega > \omega_b; \end{cases}$$

ξ_k is the positive root of the equation below

$$g\xi \tan h\xi = -\omega^2, \quad \zeta_k = \sqrt{\xi_k^2 + m^2}, \quad k \geq 1.$$

It should be mentioned for the rigid bottom, $w_0 = 0$, our spectral problem has only continuous spectrum beginning with the cut-off frequency ω_b, where $\omega_b = \sqrt{gm \tanh mH}$.

The solution of the problem (7)-(9) has the following form

$$\varphi(x,y) = -\mathrm{i}\omega w_0 \int_{-a}^{a} G(|x-\eta|, y, \omega) d\eta. \tag{13}$$

Setting m, k, M, D arbitrary parameters, on which the spectral parameter ω (fundamental frequency) depends, and substituting (13) into (10) we can have the following transcendental equation to determine the fundamental frequencies

$$\tilde{k}_m - M\omega^2 = M_\rho(\omega)\omega^2, \quad M_\rho(\omega) = -\rho \int\int_{-a}^{a} G(|x-\eta|, 0, \omega) d\eta dx. \tag{14}$$

Analyzing the frequency equation (14) one can come to the following results.

- For $\omega < \omega_b$,

the unique fundamental frequency ω_{I} *exists, if and only if* $Dm^4 + k > 2a\rho g$;

the following estimate $0 < \omega_{\mathrm{I}} < \min\left\{\omega_b, \sqrt{\tilde{k}_m/M}\right\}$. *holds for the fundamental frequency and the corresponding trapped mode is*

$$\varphi(x,y) = -2\mathrm{i}\omega_{\mathrm{I}} w_0 \begin{cases} \sum_{k=0}^{\infty} B_k(y,\omega_{\mathrm{I}})[1 - e^{-\zeta_k a}\cosh \zeta_k x], & |x| < a, \\ \sum_{k=0}^{\infty} B_k(y,\omega_{\mathrm{I}}) e^{-\zeta_k |x|}, & |x| > a; \end{cases}$$

if $ma >> 1$, *the fundamental frequency can be given by the following approximation*

$$\omega_{\mathrm{I}}^2 \approx \frac{\tilde{k}_m}{M + \frac{2a\rho}{m}\coth mH}.$$

- For $\omega = \omega_b$, the problem has only trivial solution $\varphi = 0$.
- For $\omega > \omega_b$, the Green function is the complex, with a consequent formation of surface travelling waves carrying the energy to infinity. The condition (11) is fulfilled when $\omega = \omega_{\mathrm{II}n}$,

$$\omega_{\mathrm{II}n}^2 = g\sqrt{m^2 + \pi^2 n^2/a^2} \tanh\left(H\sqrt{m^2 + \pi^2 n^2/a^2}\right), \quad n \geq 0. \tag{15}$$

The frequency $\omega_{\mathrm{II}n}$ *(15) is fundamental one, if and only if the generalized rigidity* k_m *is defined in terms of another parameters as* $k_m = (M + M_\rho(\omega_{\mathrm{II}n}))\omega_{\mathrm{II}n} + 2a\rho g$, *and the corresponding trapped mode is expressed by*

$$\varphi(x,y) = -2\mathrm{i}\omega_{\mathrm{II}n} w_0 \begin{cases} B_0(y,\omega_{\mathrm{II}n})[1 - \cos\zeta_0 a \cos\zeta_0 x] + \sum_{k=1}^{\infty} B_k(y,\omega_{\mathrm{II}n})[1 - e^{-\zeta_k a}\cosh\zeta_k x], & |x| < a, \\ \sum_{k=1}^{\infty} B_k(y,\omega_{\mathrm{II}n}) e^{-\zeta_k |x|}, & |x| > a. \end{cases}$$

4. The case of a rectangular trench with an elastic bottom

Let us consider the channel with an uneven bottom. We divide the infinite domain W into to parts $W = \overline{W^{(+)}} \cup \overline{W^{(-)}}$ (see Fig.1). We give equivalent formulation of our problem (7)-(10) that is set only in the bounded domain $W^{(-)}$

$$\nabla^2\varphi = m^2\varphi \ \text{ in } \ W^{(-)}, \quad \varphi = B_1\varphi \ \text{ when } \ |x| < a, \ y = 0, \tag{16}$$

$$\varphi_y = -\lambda B_2\varphi \ \text{ when } \ |x| < a, \ y = -h, \quad \varphi_x = 0 \ \text{ when } \ x = \pm a. \tag{17}$$

where $B_1\varphi = \int_{-a}^{a} \varphi_y(\eta, 0) G(x-\eta, 0, \omega) d\eta$, $B_2\varphi = \dfrac{\omega^2}{\tilde{k}_m/M - \omega^2} \int_{-a}^{a} \varphi(x, -h) dx$ and G is the

Green function (12). Further the solution of the problem (16)-(17) will be sought for $\omega < \omega_b$. We tread the parameter $\lambda = \frac{\rho}{M}$ as a spectral parameter. The potential φ is expressed by the unknown function ψ in the following way

$$\varphi(x,y) = -\lambda\psi(x,y)B_2\varphi, \tag{18}$$

where ψ is the solution of the following problem

$$\nabla^2\psi = m^2\psi \text{ in } W^{(-)}, \quad \psi = B_1\psi \text{ when } |x| < a, \ y = 0, \tag{19}$$

$$\psi_y = 1 \text{ when } |x| < a, \ y = -h, \quad \psi_x = 0 \text{ when } x = \pm a. \tag{20}$$

Separating of variables in the problem (19)-(20) yields the nonhomogeneous infinite system of algebraic equations. It can be shown that the system has the unique bounded solution and the solution of the problem (19)-(20) exists and is unique. Integrating (18) with respect to x from $-a$ to a we determine the spectral parameter

$$\frac{1}{\lambda} = \frac{\omega^2}{\omega^2 - \tilde{k}_m/M}\int_{-a}^{a}\psi(x,-h)dx. \tag{21}$$

It can be shown by analysis of the problem (19)-(20) that $\int_{-a}^{a}\psi(x,-h)dx < 0$. Then we obtain the following results

- *For $\omega < \omega_b$, if the frequency satisfying the inequalities $0 < \omega < \min\left\{\omega_b, \sqrt{\tilde{k}_m/M}\right\}$. there exists only one spectral parameter λ and the corresponding trapped mode is given by (18). If $ma >> 1$, the fundamental frequency can be expressed by*

$$\frac{1}{\lambda} \approx \frac{\tilde{k}_m}{\rho\omega^2} - 2a\frac{\coth mH + \tanh mh}{m(1 + \tanh mh \coth mH)}$$

- *For $\omega = \omega_b$, there exists only one spectral parameter λ which is given by*

$$\frac{1}{\lambda} = \frac{1}{mg\rho}\left[k_m \coth mH + 2ag\rho(\coth mH - \coth mh)\right]$$

and the corresponding eigenfunction is $\varphi(x,y) = 0$ in $W^{(+)}$ and $\varphi(x,y) = \varphi(y)$ in $W^{(-)}$. Then we have the interesting phenomenon. The fluid oscillations occur only in the bounded domain, $W^{(-)}$, and the fluid is rest in the infinite domain $W^{(+)}$.

5. Conclusion

The possibility of the trapped modes phenomenon in the finite depth channel with elastic inclusion on the bottom has been demonstrated. Two cases of the geometry of the rigid bottom for which no trapped modes exist has been considered. We have obtained the conditions of the existence trapped modes for the different ranges of the frequency.

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