Rapidly convergent representations for free-surface Green's functions

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Introduction

There are two key ingredients to the derivations of the formulas in this paper. The first is the observation that solutions to Poisson's equation are related to solutions of the heat equation. Thus if

$$\nabla^2 u = f \qquad \text{in } \Omega \tag{1}$$

and

$$\nabla^2 v = v_t \qquad \text{in } \Omega, \tag{2}$$

$$v = -f \qquad \text{at } t = 0, \tag{3}$$

with u and v satisfying the same time-independent boundary conditions on $\partial\Omega$, then

$$u = \int_0^\infty v \,\mathrm{d}t, \tag{4}$$

provided this integral exists.

The other important step in the derivations below is to find two complementary representations for v, v_1 and v_2, the first of which is easy to calculate for small values of t, the latter being easily evaluated for large t. We can then introduce an arbitrary positive parameter a and hence obtain a one-parameter family of formulas for u in the form

$$u = \int_0^a v_1 \,\mathrm{d}t + \int_a^\infty v_2 \,\mathrm{d}t. \tag{5}$$

These ideas were used by Strain (1992) to derive rapidly convergent series for the Green's function associated with Laplace's equation in an n-dimensional cube.

In this work we will apply these ideas in order to derive rapidly convergent expressions for Green's functions associated with water-wave problems in which the water depth is constant. One consequence of the fact that the domain Ω is unbounded is that the integral in (4) does not exist and the above procedure has to be modified slightly. Thus we choose $\tilde{v}$ so that $\int_0^\infty (v + \tilde{v}) \,\mathrm{d}t$ does exist. Then the value of this integral is $u + \tilde{u}$ where

$$\nabla^2 \tilde{u} = -\tilde{v}\Big|_{t=0}. \tag{6}$$

Provided we can solve this equation we then have

$$u = \int_0^\infty (v + \tilde{v}) \,\mathrm{d}t - \tilde{u}. \tag{7}$$

New representations for free-surface Green's functions

We will use the following definitions:

$$r = [x^2 + y^2 + (z-\zeta)^2]^{1/2}, \qquad r' = [x^2 + y^2 + (2h + z + \zeta)^2]^{1/2},$$
$$R = [x^2 + y^2]^{1/2}, \qquad \rho = [x^2 + (z-\zeta)^2]^{1/2}, \qquad \rho' = [x^2 + (2h + z + \zeta)^2]^{1/2},$$
$$\chi_n^{(1)} = 2nh - \zeta - z, \qquad \chi_n^{(2)} = 2nh - \zeta + z,$$
$$\chi_n^{(3)} = 2nh + \zeta - z, \qquad \chi_n^{(4)} = 2nh + \zeta + z.$$

The exponential integral, $E_1(x)$, the incomplete Gamma function, $\Gamma(a,x)$ and the complementary error function $\mathrm{erfc}(x)$ will also be used.

Two dimensions

We consider the two-dimensional fluid domain $-\infty < x < \infty$, $-h < z < 0$ with the undisturbed free surface being $z = 0$ so that the Green's function representing an oscillating point source at $x = 0$, $z = \zeta$ is $Re(G\exp\{-\mathrm{i}\omega t\})$ where G is the solution to

$$\nabla^2_{xz} G = \delta(x)\delta(z-\zeta) \qquad -h < z < 0, -h < \zeta < 0, \tag{8}$$
$$G_z = KG \qquad \text{on } z = 0, \tag{9}$$
$$G_z = 0 \qquad \text{on } z = -h, \tag{10}$$

and we require G to behave like outgoing waves as $|x| \to \infty$.

Numerous representations exist for this Green's function. In particular we have the eigenfunction expansion

$$G = -\sum_{m=0}^{\infty} \frac{\cos\mu_m(z+h)\cos\mu_m(\zeta+h)}{2\mu_m N_m} \mathrm{e}^{-\mu_m|x|}, \tag{11}$$

where μ_m, $m \geq 1$ are the positive solutions to $\mu_m \tan\mu_m h + K = 0$, $\mu_0 = -\mathrm{i}\mu$ where μ is the positive root of $\mu\tanh\mu h = K$ and

$$N_m = \frac{h}{2}\left(1 + \frac{\sin 2\mu_m h}{2\mu_m h}\right). \tag{12}$$

This series converges rapidly provided $|x|$ is not too small.

Following the procedure outlined in the introduction we can derive the new representation for G,

$$\begin{aligned} G = &-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\mu|x|}}{2\mu N_0}\cosh\mu(z+h)\cosh\mu(\zeta+h) - \sum_{m=0}^{\infty}\frac{\Lambda_m}{N_m}\cos\mu_m(z+h)\cos\mu_m(\zeta+h) \\ &-\frac{1}{4\pi}E_1\left(\frac{\rho^2}{a^2h^2}\right) - \frac{1}{4\pi}E_1\left(\frac{\rho'^2}{a^2h^2}\right) - \sum_{n=1}^{\infty}(-1)^n L_n, \end{aligned} \tag{13}$$

where a is an arbitrary positive parameter,

$$\Lambda_0 = -\int_0^{a^2h^2/4} \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} e^{\mu^2 t}\, dt \tag{14}$$

$$= -\frac{1}{4\pi^{1/2}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\mu}{2}\right)^{2n} \left[|x|^{2n+1}\Gamma\left(-\frac{1}{2}-n\right) - \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m!(m-n-\frac{1}{2})(ah)^{2m-2n-1}}\right], \tag{15}$$

$$\Lambda_m = \int_{a^2h^2/4}^{\infty} \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} e^{-\mu_m^2 t}\, dt \tag{16}$$

$$= \frac{1}{2\pi^{1/2}\mu_m} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{x\mu_m}{2}\right)^{2n} \Gamma\left(\frac{1}{2}-n, \frac{\mu_m^2 a^2 h^2}{4}\right), \tag{17}$$

$$L_n = \int_0^{a^2h^2/4} \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} \left(I_{n,n}(\chi_{n-1}^{(1)}) + I_{n,n}(\chi_n^{(2)}) + I_{n,n}(\chi_n^{(3)}) + I_{n,n}(\chi_{n+1}^{(4)})\right) dt \tag{18}$$

and $I_{n,n}(\chi)$ is a known function. If we set $a = 0$ in (13) we recover the eigenfunction expansion (11).

For large values of $|x|$ the integrals Λ_0 and Λ_m are best evaluated numerically, whereas for small $|x|$ the series representations can be used. The integrals L_n must be evaluated numerically but provided a is chosen small enough only L_1 is required. We note that

$$\int_0^{a^2h^2/4} \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}} I_{1,1}(\chi)\, dt =$$
$$-\frac{1}{4\pi} E_1\left(\frac{x^2+\chi^2}{a^2h^2}\right) - \frac{K e^{-K\chi}}{\pi^{1/2}} \int_0^{ah/2} e^{K^2u^2 - x^2/4u^2} \operatorname{erfc}\left(\frac{\chi}{2u} - Ku\right) du. \tag{19}$$

Both the sums in (13) converge exponentially with the parameter a controlling the relative rates of convergence of the two series. For $a = 0$ the eigenfunction expansion (11) is recovered. The second sum in (13) is exponentially localized in space and so we can think of it as representing local information whereas global low-frequency information is represented by the first sum. This type of decomposition is known as Ewald summation.

Three dimensions

Next we consider the three-dimensional problem

$$\nabla^2 G = \delta(x)\delta(y)\delta(z-\zeta) \qquad -h < z < 0, -h < \zeta < 0 \tag{20}$$

together with (9) and (10), and we require G to behave like outgoing waves as $R \to \infty$. The eigenfunction expansion for G is

$$G = -\sum_{m=0}^{\infty} \frac{K_0(\mu_m R)}{2\pi N_m} \cos\mu_m(z+h)\cos\mu_m(\zeta+h). \tag{21}$$

Computations by Newman (1985), (1992) show that when $R/h > 1/2$ this expansion is sufficient. Our new representation for G is

$$G = -\frac{i}{4N_0} H_0^{(1)}(\mu R)\cosh\mu(z+h)\cosh\mu(\zeta+h) - \sum_{m=0}^{\infty} \frac{\Lambda_m}{N_m}\cos\mu_m(z+h)\cos\mu_m(\zeta+h)$$
$$-\frac{1}{4\pi r}\operatorname{erfc}\left(\frac{r}{ah}\right) - \frac{1}{4\pi r'}\operatorname{erfc}\left(\frac{r'}{ah}\right) - \sum_{n=1}^{\infty}(-1)^n L_n, \tag{22}$$

where

$$\Lambda_0 = -\int_0^{a^2h^2/4} \frac{e^{-R^2/4t}}{4\pi t} e^{\mu^2 t}\,dt \tag{23}$$

$$= \frac{1}{4\pi}\left(\gamma + 2\ln\frac{R}{ah} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n\,n!}\left(\frac{R}{ah}\right)^{2n}\right) - \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\mu ah}{2}\right)^{2n} E_{n+1}\left(\frac{R^2}{a^2h^2}\right), \tag{24}$$

$$\Lambda_m = \int_{a^2h^2/4}^{\infty} \frac{e^{-R^2/4t}}{4\pi t} e^{-\mu_m^2 t}\,dt \tag{25}$$

$$= \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{R}{ah}\right)^{2n} E_{n+1}\left(\frac{\mu_m^2 a^2h^2}{4}\right), \tag{26}$$

$$L_n = \int_0^{a^2h^2/4} \frac{e^{-R^2/4t}}{4\pi t}\left(I_{n,n}(\chi_{n-1}^{(1)}) + I_{n,n}(\chi_n^{(2)}) + I_{n,n}(\chi_n^{(3)}) + I_{n,n}(\chi_{n+1}^{(4)})\right) dt. \tag{27}$$

If we set $a = 0$ in (22) we recover the eigenfunction expansion (21). The logarithmic singularity in Λ_0 as $R \to 0$ is, of course, exactly that required to cancel the singularity in the Hankel function. Hence, by writing $-\frac{i}{4}H_0^{(1)}(\mu R) - \frac{1}{2\pi}\ln R$ as a power series, (22) is easily computed for small R.

For the evaluation of L_1 we note that

$$\int_0^{a^2h^2/4} \frac{e^{-R^2/4t}}{4\pi t} I_{1,1}(\chi)\,dt =$$

$$-\frac{1}{4\pi\chi}\operatorname{erfc}\left(\frac{(R^2+\chi^2)^{1/2}}{ah}\right) - \frac{Ke^{-K\chi}}{2\pi}\int_0^{ah/2} e^{K^2u^2 - R^2/4u^2}\operatorname{erfc}\left(\frac{\chi}{2u} - Ku\right) du. \tag{28}$$

Discussion

New representations have been derived for the finite-depth free-surface Green's function in two and three dimensions. These representations contain an arbitrary positive parameter a which can be varied so as to achieve the optimum convergence rate for the given physical parameters. Preliminary numerical calculations suggest that the method is very efficient. Numerical results showing the relative strengths and weaknesses of these new formulas compared with other techniques for calculating these Green's functions will be shown at the workshop.

The same techniques can be used to derive formulas for other Green's functions associated with water-wave problems and these will also be discussed.

References

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