Uniqueness, trapped modes and the cut-off frequency

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Introduction

For several years many authors tried to prove that the two-dimensional, linear water-wave problem was uniquely posed at all frequencies until McIver (1996) showed that trapped modes exist for pairs of special bodies placed in the free surface. Trapped modes are defined to be non-zero solutions of the homogeneous problem which have finite energy. Their existence at a specific frequency means that the forced problem does not have a unique solution at that frequency. The question of whether or not trapped modes exist for purely submerged bodies or variable sea-bed topography is still open. Uniqueness has been proved for some geometrical configurations of bodies and topography (see McIver 1996 for a review of the literature) but recently Evans & Porter (1998) showed that trapped modes exist for submerged bodies in the presence of surface-piercing bodies.

Trapped modes are known to occur in other types of boundary value problems. A classic example is the Stokes' edge wave which is trapped above a sloping beach and propagates along the shoreline. More recently Evans, Levitin & Vassiliev (1994) proved that trapped modes exist when bodies are symmetrically placed in water wave channels or guides. Unlike the modes found by McIver (1996), both of these types of trapped modes occur at frequencies which are less than a 'cut-off' value, below which waves cannot propagate to infinity. In the terminology of spectral theory, the trapped modes occur at frequencies ('eigenvalues') which are below the bottom of the continuous spectrum for the problem and they can be shown to exist with the use of a variational principle. However, if there is no cut-off in the problem, the variational argument fails to prove the existence of trapped modes and this is one reason why the two-dimensional water wave problem is difficult to analyse.

The purpose of this work is to show how a cut-off may be artificially introduced into the two-dimensional water-wave problem and how, for a wide class of bodies and variable topography, uniqueness may be established below this cut-off. Work is currently in progress to see whether trapped modes may be shown to exist below this cut-off and whether the trapped mode found by McIver (1996) is associated with a cut-off.

A cut-off frequency for the two-dimensional problem

The velocity potential which describes the two-dimensional, small oscillations of an inviscid and incompressible fluid at angular frequency ω is given by $Re[\phi(x,y)\, e^{-i\omega t}]$ where ϕ satisfies

$$\nabla^2\phi = 0, \text{ in the fluid} \tag{1}$$

and

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \text{ on } y = 0. \tag{2}$$

Axes are chosen so that the origin is in the mean free surface and the y-axis points vertically downwards and the parameter $K = \omega^2/g$ where g is the acceleration due to gravity. In addition, no flow through any rigid surface means that

$$\frac{\partial\phi}{\partial n} = 0 \text{ on the sea-bed and any bodies.} \tag{3}$$

If trapped modes are sought then the radiation condition is replaced by

$$\phi \to 0 \text{ as } |x| \to \infty. \tag{4}$$

Uniqueness is established if the only solution to (1)-(4) is the trivial solution $\phi = 0$. Without loss of generality, ϕ may be assumed to be real because if it were complex then the real and imaginary parts would separately satisfy the governing equations and boundary conditions. To be specific the problem in which there are no bodies in the fluid but there is a variable sea-bed which lies between $x = \pm a$, as illustrated in figure 1, is studied.

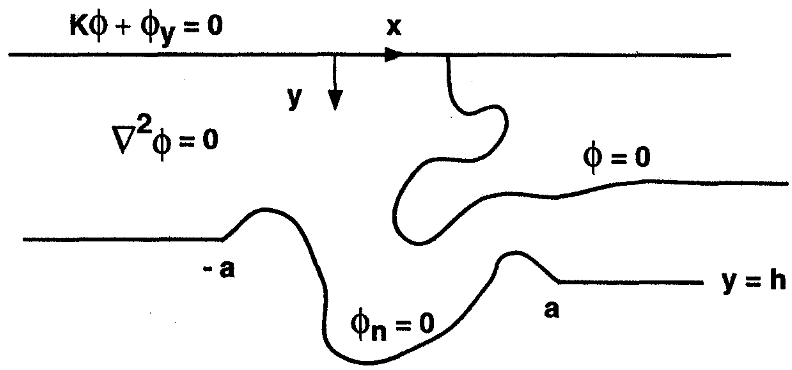

Figure 1 - Definition sketch and illustration of a nodal line

Greens theorem

$$\int_D \phi\nabla^2\psi - \psi\nabla^2\phi \, \mathrm{d}V = \int_{\partial D} \phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n} \, \mathrm{d}S \tag{5}$$

is applied to ϕ and the harmonic function

$$\psi = \sin k(x-b)\cosh k(y-h) \tag{6}$$

in the region $x \geq b \geq a$, $0 \leq y \leq h$, where h is the uniform depth of the layer in the region $x \geq a$ and kh is the real, positive root of the dispersion relation $Kh = kh \tanh kh$. In this region, both ϕ and ψ are harmonic and satisfy the same boundary conditions on the sea-bed and the free surface. As $\phi \to 0$ as $x \to \infty$ the only contribution to (5) comes from the line $x = b$ and yields

$$\int_0^h \phi(b,y)\cosh k(y-h) \, \mathrm{d}y = 0. \tag{7}$$

The function $\cosh k(y-h)$ is strictly positive and as $\phi(b,y)$ is a continuous function of y,

$$\phi(b, y_0) = 0 \tag{8}$$

for some $y_0(b)$ such that $0 < y_0 < h$. A value of y_0 may be found for every $b \geq a$ and so by continuity, there is a nodal line on which $\phi = 0$ in the interior of the fluid which extends to infinity in the region $x \geq a$. Moreover in $x \geq a$, ϕ may be represented by an eigenfunction expansion, namely

$$\phi = \sum_{n=1}^{\infty} a_n \cos k_n(y-h)e^{-k_n(x-a)} \tag{9}$$

where $\{k_n h\}$ is the monotonically increasing sequence of positive roots of the dispersion relation $Kh = -k_n h \tan k_n h$. If ϕ is not identically equal to zero then for $x \gg a$ it is dominated by the first non-zero term in this series, so for some j

$$\phi = a_j \cos k_j(y-h)e^{-k_j(x-a)} + O(e^{-k_{j+1}x}) \text{ as } x \to \infty \tag{10}$$

and so there is a nodal line which asymptotes to the horizontal line $y = d$ as $x \to \infty$, where $k_j d$ is the smallest root of the equation $\cos k_j(y-h) = 0$. Furthermore, if the potential does correspond to a trapped mode, the other end of this line cannot lie on the sea-bed or go to either infinity. If it did then there would be a region in the fluid which was open to infinity and partially surrounded by lines on which either ϕ or its normal derivative were zero and a simple application of the divergence theorem would mean that $\phi = 0$ everywhere within that region and, by analytic continuation, $\phi = 0$ everywhere in the fluid. Thus if ϕ represents a trapped mode there is a nodal line which asymptotes to the line $y = d$ as $x \to \infty$ and whose other end lies on the free surface, as illustrated in figure 1. Although the precise position of the line is unknown, it defines the lower boundary of a subregion of the fluid contained between it and the free surface. In the next section it will be shown that there is a cut-off for this new region, below which waves cannot propagate to infinity and uniqueness will be established for $Kh_{\max} \leq 1$ where $h_{\max}$ is the maximum depth of the fluid.

Uniqueness below the cut-off

The velocity potential for waves which propagate in a fluid layer of uniform depth d and which satisfies the condition $\phi = 0$ on the lower boundary, is given by

$$\phi = \sinh k(y-d)e^{\pm ikx}, \tag{11}$$

where, to satisfy the free surface condition (2), kd is a root of the dispersion relation

$$Kd = kd \coth kd. \tag{12}$$

By examining the graph of $y = x \coth x$ it is straightforward to show that there are no real roots of (12) if $Kd < 1$. Thus there is a cut-off frequency below which waves cannot propagate in a uniform layer and satisfy $\phi = 0$ on the lower boundary.

The region D is defined to be the region contained between the nodal line and the free surface and the coordinate axes are redefined so that the origin is at the intersection of the nodal line and the free surface. Integration down a vertical line from any point b on the free surface of this new region gives

$$\phi(b,0) = -\int_0^{d(b)} \frac{\partial \phi}{\partial y}(b,y)\,\mathrm{d}y, \tag{13}$$

where $d(b)$ is the smallest value of y such that the point $(b, d(b))$ lies on the nodal line. (If there is only one such value then $y = d(b)$ is the equation of the nodal line.) By squaring (13) and using the Cauchy-Schwarz inequality it may be shown that

$$\phi^2(b,0) \leq \left[\int_0^{d(b)} 1^2\,\mathrm{d}y\right]\left[\int_0^{d(b)} \left(\frac{\partial \phi}{\partial y}\right)^2 \mathrm{d}y\right] \leq d_{\max} \int_0^{d(b)} \left(\frac{\partial \phi}{\partial y}\right)^2 \mathrm{d}y, \tag{14}$$

where $d_{\max}$ is the maximum depth of the nodal line. An application of the divergence theorem with the use of (4) and (14) gives

$$\int_D (\nabla\phi)^2 \mathrm{d}V = K\int_0^\infty \phi^2(x,0)\,\mathrm{d}x \le Kd_{\max}\int_0^\infty\int_0^{d(x)}\left(\frac{\partial\phi}{\partial y}\right)^2 \mathrm{d}y\,\mathrm{d}x \le Kd_{\max}\int_D(\nabla\phi)^2\,\mathrm{d}V. \qquad (15)$$

If $Kd_{\max} < 1$ the inequality in (15) is only satisfied if $(\nabla\phi)^2$ is identically equal to zero which means that ϕ is a constant and this constant must be zero from the nodal line condition. So there are no trapped modes in the subregion for $Kd_{\max} < 1$. As $d_{\max} < h_{\max}$, the maximum depth of the fluid, there are no trapped modes in the subregion and by analytic continuation, the whole fluid, if $Kh_{\max} \le 1$.

Uniqueness for bodies and variable topography

The analysis of the previous section may be extended to the case where there are a finite number of nonbulbous, surface-piercing bodies in a fluid layer of variable depth. In this case, the nodal line may end on one of the bodies instead of the free surface. However, the nodal line and a portion of the body would still define the lower boundary of a subregion of the fluid and if the body is nonbulbous, vertical lines may be extended from every point on the free surface in the subregion to the nodal line and the analysis of the previous section will apply. In addition the proof of uniqueness for $Kh_{\max} \le 1$ extends to the case where there is a single submerged or surface-piercing body of any shape. This is because there is also a nodal line which asymptotes to the line y =const as $x \to -\infty$ and it is impossible for both nodal lines to end on the body unless the potential is identically equal to zero. Thus, at least one of the nodal lines must end on the free surface and this defines a subregion of the fluid in which the argument of the previous section may be applied.

Conclusion

Uniqueness of potential for the two-dimensional, linear boundary value problem for water waves has been proved for general sea-bed topographies for $Kh_{\max} \le 1$. The result has also been extended to prove uniqueness for the same range of frequencies when there are any finite number of nonbulbous, surface-piercing bodies in the fluid or a single submerged or surface-piercing body of any shape. The numerical evidence is that the nodal line for the trapped mode potential obtained by McIver (1996) ends on one of the bodies. However, because the bodies found are bulbous it is not possible to extend vertical lines from every point on the free surface in the subregion to the nodal line and so there is no contradiction between the existence of this mode and the uniqueness results generated in this paper.

References

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