On the completeness of eigenfunction expansions in water-wave problems

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1 Introduction

The method of eigenfunction expansions is a popular tool for the solution of the linear water-wave problem in constant depth water. The key result is that there exists a complete set of orthonormal vertical eigenfunctions so that any 'reasonable' function of the vertical coordinate may be expanded in terms of this complete set. This result comes from the theory of self-adjoint linear differential operators which is used extensively in many engineering applications of mathematics.

There are a number of problems involving wave interaction with a permeable breakwater or a perforated barrier for which the vertical eigenvalue problem is no longer self adjoint. A consequence of this is that the familiar theorems required to construct an eigenfunction expansion no longer apply and the 'obvious' eigenfunctions may not form a complete set. Perhaps the simplest problem of this type is examined in detail here, but first of all some aspects of the 'standard' water-wave problem are recalled.

2 The water-wave problem

Consider the linear water-wave problem for time-harmonic motion of angular frequency ω in a region of constant depth h, and let y be the vertical coordinate with origin in the free surface and directed upwards. An attempt to find a solution in terms of vertical eigenfunctions leads to consideration of the differential equation

$$T\chi \equiv -\frac{\mathrm{d}^2\chi}{\mathrm{d}y^2} = \lambda\chi \quad \text{for} \quad -h < y < 0 \tag{1}$$

together with the boundary conditions

$$\frac{\mathrm{d}\chi}{\mathrm{d}y} = 0 \quad \text{on} \quad y = -h \quad \text{and} \quad \frac{\mathrm{d}\chi}{\mathrm{d}y} = K\chi \quad \text{on} \quad y = 0, \tag{2}$$

where K is the real number ω^2/g and g is the acceleration due to gravity. It is well known that the solutions of this problem are of the form

$$\chi = \cos k(y+h) \tag{3}$$

where $k = \lambda^{1/2}$ is a root of the dispersion relation

$$K = -k \tan kh. \tag{4}$$

This dispersion relation has two purely imaginary roots $k = \pm k_0$ and an infinity of purely real roots $\{k = \pm k_m; m = 1, 2, \ldots\}$. The set of vertical eigenfunctions

$$\chi_m = \frac{\cos k_m(y+h)}{N_m}, \quad m = 0, 1, 2, \ldots, \tag{5}$$

with

$$N_m^2 = \frac{1}{2}\left(1 + \frac{\sin 2k_m h}{2k_m h}\right), \tag{6}$$

form a complete orthonormal set satisfying

$$\frac{1}{h}\int_{-h}^{0} \chi_m(y)\chi_n(y)\,\mathrm{d}y = \delta_{mn} \tag{7}$$

where δ_{mn} is the Kronecker delta.

It is convenient to introduce an inner product notation. Let u and v be any two functions that are square-integrable over the depth and define their inner product by

$$\langle u, v \rangle = \frac{1}{h} \int_{-h}^{0} u\bar{v} \, \mathrm{d}y \tag{8}$$

where the over bar denotes complex conjugate. In this notation, the orthogonality condition (7) is

$$\langle \chi_m, \chi_n \rangle = \delta_{mn}. \tag{9}$$

By the expansion theorem, any function f that is square integrable over the depth may be written

$$f = \sum_{m=0}^{\infty} \langle f, \chi_m \rangle \chi_m. \tag{10}$$

3 Wave motion in a permeable breakwater

A model for time-harmonic motion in a permeable breakwater[1] leads again to the consideration of the boundary-value problem (1–2) but with K now a complex number. This problem has been examined in some detail by Dalrymple, Losada & Martin[2]. In particular, they note that for certain values of the complex parameter K there are double roots of the dispersion relation (4) and, for these values of K, the eigenfunctions (5) no longer form a complete set. Dalrymple *et al.* obtain the missing eigenfunctions by an indirect argument based on the Green's function for the particular water-wave problem under consideration. Here, the problem is re-examined from the point of view of the general theory of non-self-adjoint linear differential operators.

An operator T is self adjoint if, for all suitable functions u and v,

$$\langle Tu, v \rangle = \langle u, Tv \rangle. \tag{11}$$

Integration by parts shows that this relation is satisfied by the operator defined by (1–2) provided K is real. The corresponding breakwater problem, where K is complex, is not self adjoint and the familiar expansion theorems do not apply.

Fortunately, this particular problem falls into a class discussed in Chapter 12 of the text by Coddington & Levinson[3]. The eigenvalues of the problem (1–2) are given by $\lambda = k^2$, where k is a, now complex, root of the dispersion relation (4). Let C_n be a closed contour in the complex λ plane which encircles in an anticlockwise direction the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, arranged in order of increasing modulus. The expansion theorem[3] says that, for suitable functions f,

$$f(y) = -\lim_{n\to\infty} \int_{-h}^{0} P_n(y, \eta) f(\eta) \, \mathrm{d}\eta \tag{12}$$

where

$$P_n(y, \eta) = \frac{1}{2\pi i} \int_{C_n} G(y, \eta; \lambda) \, \mathrm{d}\lambda, \tag{13}$$

G is the Green's function for the particular problem under consideration, and provided suitable convergence criteria can be established. The Green's function for the problem (1–2) is

$$G(y, \eta; \lambda) = -\frac{(k \cos ky_> + K \sin ky_>) \cos k(y_< + h)}{k(K \cos kh + k \sin kh)}, \quad k = \lambda^{1/2}, \tag{14}$$

where

$$y_< = \min(y, \eta) \quad \text{and} \quad y_> = \max(y, \eta). \tag{15}$$

This Green's function has poles at values of λ corresponding to the roots of the dispersion relation (4) so that, by the residue theorem,

$$P_n(y,\eta) = \sum_{m=1}^{n} R_m(y,\eta) \tag{16}$$

where R_m is the residue of G at $\lambda = \lambda_m$. If the eigenvalues are known then the residues at the poles of the Green's function can be calculated and the form of the general expansion found. There are two difficulties with this, one numerical and one theoretical.

The numerical difficulty is in locating the eigenvalues in the complex plane. In the case of real K the roots of the dispersion relation lie on either the real or imaginary axis in the complex k plane and are therefore easily located. For complex K, Dalymple *et al.*[2] used a numerical scheme in which the roots are tracked individually as the imaginary part of K is increased from zero. Some new results have been obtained that should allow a more direct computation of these roots.

The theoretical problem is that for isolated values of K there is a double root of the disperion relation and therefore a double pole of the Green's function. These double roots correspond to zeros in the complex k plane of the normalisation factor N_m defined in equation (6); the corresponding value of K follows from the dispersion relation (4). For almost all values of K there are no double roots.

The residue of the Green's function for a pole of order p_m at $\lambda = \lambda_m$ is readily evaluated and may be written

$$R_m(y,\eta) = \sum_{q=1}^{p_m} \overline{\psi}_{m,p_m-q+1}(\eta)\chi_{m,q}(y). \tag{17}$$

For the case of a simple pole, $p_m = 1$, the so-called 'generalised eigenfunctions' are given by

$$\chi_{m,1} = \frac{\cos k_m(y+h)}{N_m} \quad \text{and} \quad \psi_{m,1} = \overline{\chi}_{m,1} \quad \text{with} \quad \langle \chi_{m,1}, \psi_{m,1} \rangle = 1. \tag{18}$$

For the case of a double pole, $p_m = 2$, the generalised eigenfunctions are

$$\chi_{m,1} = -\frac{2\cos k_m(y+h)}{\cos^2 kh} \quad \text{and} \quad \psi_{m,1} = \overline{\chi}_{m,1}, \tag{19}$$

$$\chi_{m,2} = \tfrac{1}{6}(4\sin^2 kh - 3)\cos k_m(y+h) + k_m(y+h)\sin k_m(y+h) \quad \text{and} \quad \psi_{m,2} = \overline{\chi}_{m,2}, \tag{20}$$

with

$$\langle \chi_{m,1}, \psi_{m,1} \rangle = \langle \chi_{m,2}, \psi_{m,2} \rangle = 0 \quad \text{and} \quad \langle \chi_{m,1}, \psi_{m,2} \rangle = \langle \chi_{m,2}, \psi_{m,1} \rangle = 1. \tag{21}$$

In the double-pole case, although the residue is well defined, there is a degree of arbitrariness in the choice of the generalised eigenfunctions $\{\chi_{m,q}, \psi_{m,q}; q = 1,2\}$. Generalised eigenfunctions corresponding to different eigenvalues are biorthogonal so that

$$\langle \chi_{m,q}, \psi_{n,r} \rangle = 0, \quad m \neq n. \tag{22}$$

With the above definitions, the general expansion theorem is

$$f = \sum_{m=1}^{\infty} \sum_{q=1}^{p_m} \langle f, \psi_{m,p_m-q+1} \rangle \chi_{m,q} \tag{23}$$

For real K, all poles of the Green's function are simple and $\psi_{m,1} = \chi_{m,1} \equiv \chi_m$ so that (23) reduces to (10), after a suitable relabelling of the eigenfunctions.

4 Solutions of Laplace's equation

The expansion theorem (23) may be used to find solutions of water wave problems. For example, suppose that a solution $\phi(x,y)$ of Laplace's equation is required satisfying the boundary condiitons

$$\frac{\partial\phi}{\partial y}=0 \quad \text{on} \quad y=-h \quad \text{and} \quad \frac{\partial\phi}{\partial y}=K\phi \quad \text{on} \quad y=0. \tag{24}$$

The solution is sought in the form

$$\phi(x,y)=\sum_{m=1}^{\infty}\sum_{q=1}^{p_m}C_{m,q}(x)\chi_{m,q}(y) \tag{25}$$

which satisfies the Laplace equation provided

$$\sum_{m=1}^{\infty}\sum_{q=1}^{p_m}\left\{C''_{m,q}(x)\chi_{m,q}(y)+C_{m,q}(x)\chi''_{m,q}(y)\right\}=0. \tag{26}$$

Now

$$\chi''_{m,1}=-k_m^2\chi_{m,1} \quad \text{and} \quad \chi''_{m,2}=-k_m^2\chi_{m,2}-k^2\cos^2 kh\,\chi_{m,1} \tag{27}$$

so that (22) may be used to isolate terms corresponding to distinct eigenvalues. For a simple pole

$$C''_{m,1}-k_m^2C_{m,1}=0 \quad \text{and so} \quad C_{m,1}(x)=\alpha_m\,\mathrm{e}^{k_m x}+\beta_m\,\mathrm{e}^{-k_m x}. \tag{28}$$

For a double pole, application of the biothogonality properties (21) yields

$$C''_{m,2}-k_m^2C_{m,2}=0 \quad \text{and} \quad C''_{m,1}-k_m^2C_{m,1}=k^2\cos^2 kh\,C_{m,2} \tag{29}$$

which have solutions

$$C_{m,2}(x)=\gamma_m\,\mathrm{e}^{k_m x}+\delta_m\,\mathrm{e}^{-k_m x} \tag{30}$$

and

$$C_{m,1}(x)=\alpha_m\,\mathrm{e}^{k_m x}+\beta_m\,\mathrm{e}^{-k_m x}+\tfrac{1}{2}kx\cos^2 kh\left(\gamma_m\,\mathrm{e}^{k_m x}-\delta_m\,\mathrm{e}^{-k_m x}\right). \tag{31}$$

5 Conclusion

This work is concerned with a simple model for the propagation of water waves in a porous medium. The model has been extended to a two-layer flow by Yu & Chwang[4] and the problem is again not self adjoint. This modified problem involves additional matching conditions at an intermediate depth and the theorems given by Coddington & Levinson[3], and others, do not apply to this case. Thus, it is not clear that the expansion theorem is valid even when there are no double roots of the dispersion relation. This and other models are currently under further investigation.

References

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