# A procedure to remove secularity in third-order numerical wave tanks

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## Introduction

Many 2D numerical wave tanks have been developed worldwide. Most of them tackle the fully nonlinear problem, but some are based on the Stokes expansion procedure $\Phi = \epsilon\,\Phi^{(1)} + \epsilon^2\,\Phi^{(2)} + \ldots$, and are restricted to first (linear) or second-order effects. At the eleventh Workshop in Hamburg Büchmann presented a code with a third-order extension (Büchmann, 96). A similar model has been developed more recently by Stassen (Stassen et al, 1998).
In Büchmann and Stassen's codes successive boundary value problems are solved at orders $i = 1, 2, 3$, with the free surface conditions given as (at $y = 0$)

$$\Phi_t^{(i)} + g\,\eta^{(i)} = f^{(i)} \tag{1}$$

$$\eta_t^{(i)} - \Phi_y^{(i)} = h^{(i)} \tag{2}$$

where $f^{(i)}$ and $h^{(i)}$ are zero at order $i = 1$ and depend on the solution(s) obtained at the previous order(s) for $i = 2, 3$.

When a regular wave is being produced in the numerical tank, a problem that has been observed is that the third-order component to the wave elevation, associated with $\Phi^{(3)}$, tends to increase steadily in amplitude as the wave travels down the tank. As a result when $\eta = \epsilon\,\eta^{(1)} + \epsilon^2\,\eta^{(2)} + \epsilon^3\,\eta^{(3)}$ is being recomposed at a finite value of $\epsilon$ (for comparison with experimental results for instance), the third-order term $\epsilon^3\,\eta^{(3)}$ gradually overruns the second-order and first-order ones, invalidating the perturbation procedure. This is illustrated in figure 2.

This phenomenon is due to secularity. In the frequency domain the remedy is well known and consists in slightly modifying the wave number, the frequency being imposed by the wave maker motion. In deep water regular waves the wave number correction is simply $\Delta k = -\epsilon^2\,k$, $k$ being the wave number $\omega^2/g$.
In the time domain, with the wave front gradually advancing over still water (and the generated waves not necessarily being regular), a different procedure must be sought for. A possible one consists in stretching the coordinate system, as is proposed below.

## Theory

We consider two coordinate systems $(x, y)$ and $(X, Y)$, $(x, y)$ corresponding to the physical domain, and $(X, Y)$ to the computational domain. Both are centered at the free surface wave maker intersection.
The mapping between the computational domain and the physical domain is given by

$$x = X + \epsilon^2\,P(X, Y, t) \tag{3}$$

$$y = Y + \epsilon^2\,Q(X, Y, t) \tag{4}$$

with the following restrictions on $P$ and $Q$:

$$\nabla P = O(1) \qquad \nabla Q = O(1) \qquad P(0,Y,t) = 0 \qquad Q(0,0,t) = 0 \qquad k\,Q(X,0,t) = O(1)$$

$k\,P(X,Y,t)$ being unrestricted.
For the sake of conveniency we will also assume that the waterdepth $h$ is rather shallow, or $k\,Q(X,-h,t) = O(1)$, but the problem can be worked out without this assumption. (As a matter of fact it is even simpler when the waterdepth is infinite).
As a result the boundaries of the physical domain correspond to the following curves in the computational domain:

$$\begin{array}{lll} x = 0 & \text{(wavemaker)} & \to X = 0 \\ y = 0 & \text{(free surface)} & \to Y = -\epsilon^2\, Q(X,0,t) \\ y = -h & \text{(bottom)} & \to Y = -h - \epsilon^2\, Q(X,-h,t) \end{array}$$

The following step is to formulate the BVP satisfied by $\Phi(x,y,t)$ in the computational domain $(X,Y)$. Partial derivatives are transformed by

$$\frac{\partial}{\partial x} = (1-\epsilon^2 P_X)\frac{\partial}{\partial X} - \epsilon^2 Q_X \frac{\partial}{\partial Y} \tag{5}$$

$$\frac{\partial}{\partial y} = (1-\epsilon^2 Q_Y)\frac{\partial}{\partial Y} - \epsilon^2 P_Y \frac{\partial}{\partial X} \tag{6}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \epsilon^2 P_t \frac{\partial}{\partial X} - \epsilon^2 Q_t \frac{\partial}{\partial Y} \tag{7}$$

the time derivative being the eulerian one, at $x$ and $y$ fixed.

At orders 1 and 2 the BVP's are unchanged. At order 3 the Laplace equation for $\Phi^{(3)}$ is maintained provided $P$ and $Q$ verify the Cauchy-Riemann conditions $P_X = Q_Y$, $P_Y = -Q_X$, or $P(X,Y,t) + i\,Q(X,Y,t) = f(X + i\,Y, t)$.
The boundary conditions on the wave maker, bottom and free surface are transformed as follows.

**Wave maker**
For the sake of simplicity we assume the wave maker to be vertical at $x = 0$. The no-flow condition in the physical domain is of the type

$$\Phi_x^{(3)}(0,y,t) = g^{(3)}(y,t)$$

with $g^{(3)}$ depending on the solutions at orders 1 and 2.
The condition in the computational domain is then

$$\Phi_X^{(3)}(0,Y,t) = g^{(3)}(Y,t) + P_X\,\Phi_X^{(1)} + Q_X\,\Phi_Y^{(1)} = g^{(3)}(Y,t) + P_X\,\Phi_X^{(1)} \tag{8}$$

since $Q_X = P_Y \equiv 0$ at $x = X = 0$.

**Bottom** $(y = -h)$
Similarly the no-flow condition $\Phi_y^{(3)}(x,-h,t) = 0$ becomes

$$\Phi_Y^{(3)} - Q_Y\,\Phi_Y^{(1)} - P_Y\,\Phi_X^{(1)} - Q\,\Phi_{YY}^{(1)} = 0$$

at $Y = -h$, or

$$\Phi_Y^{(3)} = -\frac{\partial}{\partial X}(Q\,\Phi_X^{(1)}) \tag{9}$$

**Free surface** ($y = 0$)
The new conditions at $Y = 0$ are

$$\Phi_t^{(3)} - P_t\,\Phi_X^{(1)} - Q_t\,\Phi_Y^{(1)} - Q\,\Phi_{tY}^{(1)} + g\,\eta^{(3)} = f^{(3)}(X,t) \qquad (10)$$

$$\eta_t^{(3)} - P_t\,\eta_X^{(1)} - \Phi_Y^{(3)} + Q_Y\,\Phi_Y^{(1)} + P_Y\,\Phi_X^{(1)} + Q\,\Phi_{YY}^{(1)} = h^{(3)}(X,t) \qquad (11)$$

where $\eta(X,t) = \epsilon\,\eta^{(1)}(X,t) + \epsilon^2\,\eta^{(2)}(X,t) + \epsilon^3\,\eta^{(3)}(X,t)$.

Thanks to the free surface conditions verified by $\Phi^{(1)}$ and $\eta^{(1)}$, these two equations can be rewritten as

$$\frac{\partial}{\partial t}\,[\Phi^{(3)} - P\,\Phi_X^{(1)} - Q\,\Phi_Y^{(1)}] + g\,[\eta^{(3)} - P\,\eta_X^{(1)}] = f^{(3)} \qquad (12)$$

$$\frac{\partial}{\partial t}\,[\eta^{(3)} - P\,\eta_X^{(1)}] - \frac{\partial}{\partial Y}\,[\Phi^{(3)} - P\,\Phi_X^{(1)} - Q\,\Phi_Y^{(1)}] = h^{(3)} \qquad (13)$$

that is the same equations as in the secular case are obtained with $\Phi^{(3)}(x,0,t)$ being replaced by $\Phi^{(3)}(X,0,t) - P\,\Phi_X^{(1)} - Q\,\Phi_Y^{(1)}$ and $\eta^{(3)}(x,t)$ replaced by $\eta^{(3)}(X,t) - P\,\eta_X^{(1)}$.
This result would have been obtained readily if one had assumed both $k\,P$ and $k\,Q$ to be $O(1)$ at the free surface, through Taylor developments in $X$ and $Y$. Actually only $k\,Q = O(1)$ is required (and to be checked later).
The procedure to get rid of secularity is now straight-forward. Be $\Phi_S^{(3)}$ and $\eta_S^{(3)}$ the (secular) solutions obtained when $P = Q = 0$. Then $\eta_S^{(3)}$ contains a secular component at the same spatial frequencies as $\eta^{(1)}$. This suggests to take $P(X,0,t)$ equal to the slowly-varying part (in $X$ and $t$) of $-\eta_S^{(3)}/\eta_X^{(1)}$. Then, hopefully, $\eta^{(3)}$ will reduce to the expected small, high frequency components.
Once $P$ has been thus determined on the free surface, $P$ and $Q$ are obtained in the whole computational domain through

$$P + i\,Q\,(Z) = \frac{i}{\pi}\int_{-\infty}^{\infty}\left[\frac{P(\zeta,0,t)}{\zeta - Z} - \frac{P(\zeta,0,t)}{\zeta}\right]d\zeta \qquad (14)$$

with $Z = X + i\,Y$ and $P(-\zeta,0,t) = -P(\zeta,0,t)$. Then the modified boundary conditions, at the wave maker and on the bottom, can be accounted for.

Figures 1 through 4 show some preliminary results relating to experiments carried out in the towing tank of Ecole Centrale de Nantes. The length of the tank is 63 m, the waterdepth 2.8 m, the wave period 2.1 s and the wave amplitude 0.12 m.
Figure 1, 2 and 3 show, 35 s after the wavemaker got started, values of $\eta_X^{(1)}$, $\eta_S^{(3)}$ and $-\eta_S^{(3)}/\eta_X^{(1)}$ along the tank. Figure 4 shows, at different instants, values obtained for $P$ at the free surface, through low-pass filtering. All these results are dimensional, corresponding to $\epsilon = ka = 0.11$. It can be seen that $P$, which is nothing but the distance the first-order wave profile must be shifted forward, slowly adjusts to the steady state solution $P = k^2a^2\,X$ as the wave system gets established in the tank.

## References

BÜCHMANN, B. (1996). 'A 2-D numerical wave flume based on a third order boundary element model', Proc. 11th Int. Workshop Water Waves & Floating Bodies, Hamburg.
STASSEN, Y., LE BOULLUEC M. & MOLIN, B. (1998). 'A high order boundary element model for 2D wave tank simulation', ISOPE.

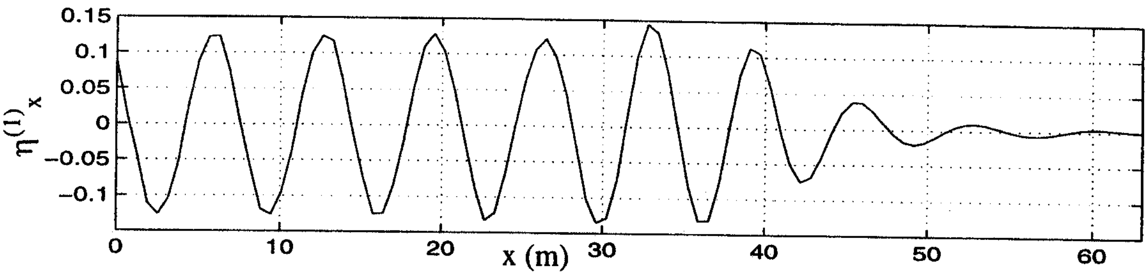


***figure 1 :*** $\eta^{(1)}_x$ *along the tank at t=35 s.*

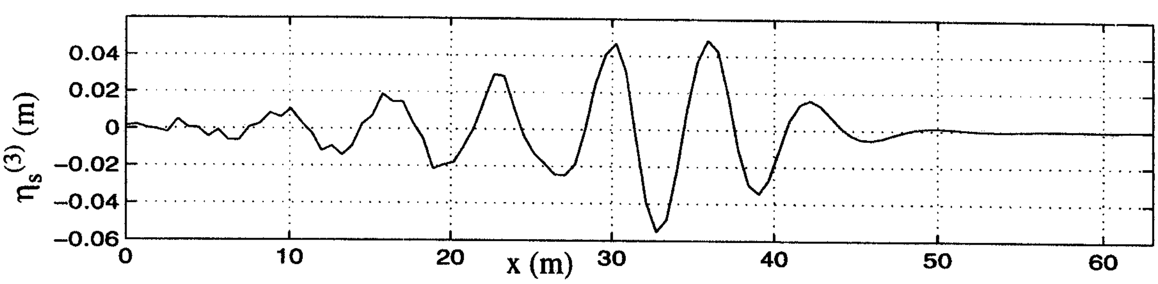


***figure 2 :*** $\eta_s^{(3)}$ *along the tank at t=35 s.*

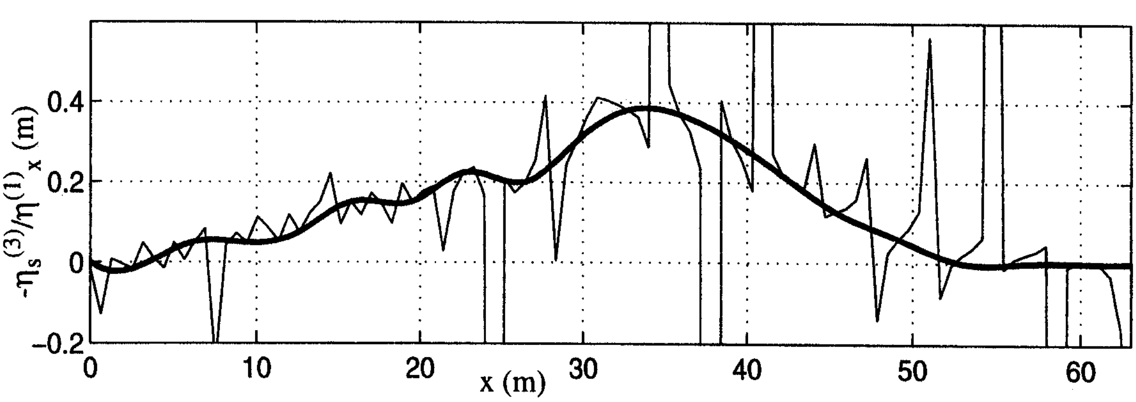


***figure 3 :*** $-\eta_s^{(3)}/\eta^{(1)}_x$ *along the tank at t=35 s.*
*The bold line shows results from low-pass filtering.*

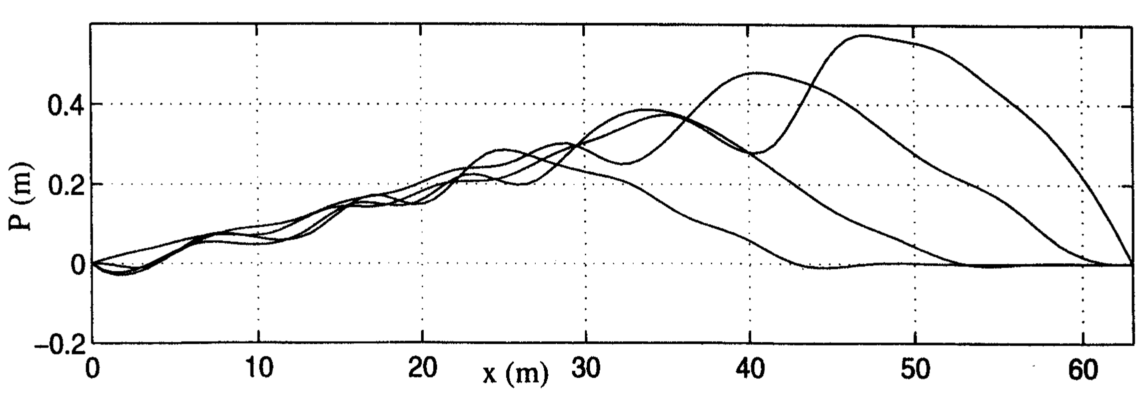


***figure 4 :*** *P obtained from low-pass filtering at t=30 s, t=35 s, t=40 s and t=45 s.*