# Some Problems of Hydroelastic Behavior of a Floating Thin Plate in Shallow Water Waves

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## 1. Introduction

Analysis of the vibration of a large but thin floating plate, a conceptual configuration of floating airpors, when it is modeled as a membrane sheet of small bending rigidity on the water surface, is extremely simplified in the framework of linear shallow water theory. There is no reason that we do not use the linear shallow water theory to discuss the behaviors of floating airports in waves. The shallow water approximation is rather more realistic; very large structures like the floating airports are supposed to be located not offshore but near-shore. Consequently the horizontal size of them and wave length of our concern are very large compared with the water depth.

In this report we present some examples of analysis of hydroelastic response of a thin plate to waves in shallow water theory. Essential idea of formulating the boundary value problem to determine the plate deflection and the fluid flow is not different from that we presented in the papers (Ohkusu & Nanba (1996), (1997)) at 11th and 12th Workshop; the draft of the plate is assumed very small and the kinematic condition underneath the plate is imposed on the level of calm water surface. Equation of the plate vibration is combined with the kinematic condition to derive a quasi free surface condition for waves on the plate representing the vertical deflection of the plate. Difference is that all those formulations are carried out by the linear shallow water.

## 2. Elongate plate in head waves

Analysis of vibration of a thin plate of elongate form in waves at oblique incidence is straightforward in the linear shallow water theory. We present here the analysis not in oblique waves but in head waves. We assume the plate width $2b$ is very small compared with the length $L$ ($\varepsilon = 2b/L$). Water depth $h$ is constant and shallow compared with other length scale. $z$ axis is vertically upward and the $x - y$ plane coincides with calm water surface. The plate is on $z = 0$ surface and occupies it at $0 \leq x \leq L, -b \leq y \leq b$

Under the plate we have the kinematic condition

$$\frac{\partial \zeta}{\partial t} = -h\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi \tag{1}$$

where $\zeta$ is the deflection, the local vertical displacement of the plate and $\phi$ is the velocity potential representing the average flow from $z = 0$ to $-h$.

The kinematic condition is combined with the dynamical condition that is the equation of the vertical vibration of the plate to derive

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^3 \phi + \frac{\rho g}{D}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \frac{\rho g}{D}\frac{\omega^2}{gh}\phi = 0 \tag{2}$$

where $D$ is the bending rigidity of the plate and $\omega$ the wave frequency.

In the region of the free water surface not covered with the plate over it $\phi$ will satisfy the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \frac{\omega^2}{gh}\phi = 0 \tag{3}$$

The kinematic condition at this part is identical to the equation (1)

We consider the case that incident wave of wave number $k(=\omega/\sqrt{gh})$ are head on the plate from the direction of the positive $x$. When we assume the plate is elongate in the $x$ direction and $kb = O(1)$, the following form of the solution will be reasonable.

$$\phi(x,y) = \psi(x,y)e^{-ikx} \tag{4}$$

Sufficiently away from the front edge at $x = 0$, $\psi$ is a slowly varying function of $x$. Hereafter we suppress the time-dependent term $e^{i\omega t}$ in the formulation.

To the lowest order of approximation equation (4) in the region $|y| = O(\varepsilon^{\frac{1}{2}})$ is rewritten as

$$-2ik\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{5}$$

Substitute (4) into the plate equation (2) and retain the lowest order terms considering $\partial/\partial y = O(\varepsilon^{-1})$ on the plate, we have

$$\left(-k^2 + \frac{\partial^2}{\partial y^2}\right)^3 \psi = 0 \tag{6}$$

Solution of equation (6) is straightforward. It will be written in the form

$$\psi = [a_1(x) + a_3(x)y + a_5(x)y^2]e^{ky} + [a_2(x) + a_4(x)y + a_6(x)y^2]e^{-ky} \tag{7}$$

$\psi$ given by (7) must satisfy the edge condition representing zero shear force and zero bending moment at $y = \pm b$. The edge conditions to the same order of approximation as equation (5) are the following four linear equations of $a_j$

$$[a_1(x)P_1(y) + a_3(x)P_3(y) + a_5(x)P_5(y)]e^{ky} + [a_2(x)P_2(y) + a_4(x)P_4(y) + a_6(x)P_6(y)]e^{-ky} = 0 \quad \text{at} \quad y = \pm b \tag{8}$$

$$[a_1(x)Q_1(y) + a_3(x)Q_3(y) + a_5(x)Q_5(y)]e^{ky} + [a_2(x)Q_2(y) + a_4(x)Q_4(y) + a_6(x)Q_6(y)]^{-ky} = 0 \quad \text{at} \quad y = \pm b \tag{9}$$

Here

$$P_{1,2}(y) = k^4, P_{3,4}(y) = \pm 4k^3 + k^4 y, P_{5,6}(y) = 12k^2 \pm 8k^3 y + k^4 y^2 \tag{10}$$

$$Q_{1,2}(y) = \pm k^5, Q_{3,4}(y) = 5k^4 \pm k^5 y, Q_{5,6}(y) = \pm 20k^3 + 10k^4 y \pm k^5 y^2 \tag{11}$$

$\psi$ on the plate given by equation (7) must match with the solution of (5) on the free water surface. The solution of (5) symmetrical in $y$ for the head waves (Tuck (1965), Mei & Tuck (1980)) is

$$\psi = 1 - \frac{1+i}{2\sqrt{\pi k}} \int_0^x d\xi \frac{V(\xi)}{\sqrt{x-\xi}} e^{iky^2/2(x-\xi)} \tag{12}$$

Matching condition will be formulated following Mei & Tuck (1980). When $y$ approaches zero ( the breadth of the plate is of the order $O(\varepsilon)$ ), $\psi$ of (12) will become

$$\psi = \psi_0(x) + V(x)|y| \tag{13}$$

where

$$\psi_0(x) = 1 - \frac{1+i}{2\sqrt{\pi k}} \int_0^x d\xi \frac{V(\xi)}{\sqrt{x-\xi}} \tag{14}$$

Matching conditions are that mass flux and energy flux given by (7) and (13) must be equal respectively at $y = \pm b$. They are

$$[a_1(x) + a_3(x)y + a_5(x)y^2]e^{ky} + [a_2(x) + a_4(x)y + a_6(x)y^2]e^{-ky} = \psi_0(x) \quad \text{at} \quad y = \pm b \tag{15}$$

$$\begin{aligned}[a_1(x)k + &a_3(x)(ky+1) + a_5(x)(ky^2+2y)]e^{ky} \\ &- [a_2(x)k + a_4(x)(ky-1) + a_6(x)(ky^2-2y)]e^{-ky} = \pm V(x) \quad \text{at} \quad y = \pm b\end{aligned} \tag{16}$$

Our solution must be symmetrical in $y$ and therefore $a_1 = a_2, a_3 = -a_4, a_5 = a_6$. Four linear equations (8),(9),(15) and (16) for $y = +b$ determine four constants $a_j (j = 1, 3, 5)$ and $\psi_0(x)$, which are linear to $V(x)$ such as $a_j(x) = A_j V(x)$ and $\psi_0(x) = \alpha V(x)$. $A_j$ and $\alpha$ are independent of $x$. Their algebraic expression is lengthy and not given here. Those solutions and equation (14) give an Abel integral equation

$$\frac{1+i}{2\sqrt{\pi k}} \int_0^x d\xi \frac{V(\xi)}{\sqrt{x-\xi}} = 1 - 2V(x)[(A_1 + A_5 b^2)\cosh kb + A_3 b \sinh kb] \tag{17}$$

The solution $V(x)$ of (17) is given in the form of the complementary error function and eventually determines the deflection $\zeta$ of the plate.

### 3. Wide plate in head waves

Floating airports are generally of elongate form. Their width is, however, very large compared with the length of incident waves. The assumption of $kb = O(1)$ as employed in the previous section is not always practical. In this section we consider a plate extending from $x = 0$ to $x = +\infty$ and from $y = 0$ to $y = -\infty$; the plate occupies a quarter of the whole water plane. Other 2/3 of the plane is the free water surface. Incident waves uniform in $y$ direction of the wave number $k$ come from $x = -\infty$. Here we are concerned with the plate deflection away from the front edge at $x = 0$.

Effect of the waves propagating in the region of the free water surface $y \geq 0$ to the plate deflection is analyzed almost the same way as in the previous section. The solution is given in the form (4) and $\psi$ on the plate part is given by (7) with $a_2 = a_4 = a_6 = 0$ for the deflection to vanish far away ($y \sim -\infty$) from the edge at $y = 0$. (7) can be matched with the solution on the free water part given in the form of (12) in the same manner as in the previous section.

The incident waves will come into the plate through the front edge at ($x = 0, y \leq 0$). They are given by

$$\phi = A_0 e^{-ik_0 x} \tag{18}$$

where $k_0$ is positive one of two real roots of the equation

$$k_0^6 + \delta k_0^2 - \delta k^2 = 0 \tag{19}$$

and $A_0$ is determined by the edge conditions at $x = 0$. Other wave components are all zero because we are away from the edges and no waves come from $x = \infty$.

Though it is straightforward to have $\psi e^{-ik_0 x}$ close to the $y = 0$ edge of the plate to be matched with (18) and the waves on the free water part $y > 0$ of the form $e^{-i\sqrt{k^2-k_0^2}y}$, the matching of them are not completed. Details of the difficulty will be presented in the Workshop.

## 4. Trapped waves on the plate

Existence of eigenfrequencies will be possible which correspond to modes of waves trapped over the plate. It was suggested by Prof. Evans at 12th IWWWFB.

We assume a solution of the form

$$\phi = \psi(y)e^{i\gamma x} \tag{20}$$

as in the section **2**, while here we consider the case of $\gamma > k(=\omega^2/gh)$. Governing equation for $\psi$ under the plate of infinite length in $x$ direction and finite width in $y$ direction will be

$$\left[\left(-\gamma^2 + \frac{\partial^2}{\partial y^2}\right)^3 + \frac{\rho g}{D}\left(-\gamma^2 + \frac{\partial^2}{\partial y^2}\right) + \frac{\rho g}{D}k^2\right]\psi = 0 \tag{21}$$

A solution is

$$\psi = \sum_{j=1}^{6} b_j e^{\lambda_j y} \tag{22}$$

Here $b_j (j = 1, 2, ...6)$ are constants and $\lambda_j$ are six roots of the equation

$$(-\gamma^2 + \lambda_j)^3 + \delta(-\gamma^2 + \lambda_j^2) + \delta k^2 = 0 \tag{23}$$

On the free water part with no plate over it the equation (3) is rewritten as

$$\frac{\partial^2 \psi}{\partial y^2} + (k^2 - \gamma^2)\psi = 0 \tag{24}$$

Solutions with no progressive wave are possible with this equation:

$$\psi = Ae^{\sqrt{\gamma^2 - k^2}(y_b)} \qquad \text{at} \quad x < -b \tag{25}$$

$$\psi = Be^{-\sqrt{\gamma^2 - k^2}(y-b)} \qquad \text{at} \quad x > b \tag{26}$$

If the solutions given by equation (22) happen to match with the solutions (25) and (26) at $y = \pm b$, then they are trapped modes over the plate. Matching condition is obtained by imposing the conditions similar to (8),(9),(15) and (16) as:

$$M \cdot \mathbf{x}^T = 0 \tag{27}$$

where $\mathbf{x}$ is the vector $\mathbf{x} = (b_1, b_2, b_3, b_4, b_5, b_6, A, B)$ and $M$ is a matrix.

If the frequencies exist at which the determinant of the matrix $M$ is zero, the solutions will be the trapped modes. Simple algebraic expression of the determinant seems not possible and the frequency of zero determinant is numerically searched.

The condition of $\gamma > k(=\omega^2/gh)$ is never realized with water waves and those trapped modes, if they exist, will be induced by other causes such as some wave impact force or wind effect. Practical implication of this phenomenon with the floating airports is to be discussed in future study.

### References

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