On the Wave Field due to a Moving Two-Dimensional, Submergded Body Oscillating Near the Critical Frequency

E. Palm and J. Grue
Department of Mathematics, University of Oslo,
P.O.Box 1053 Blindern, N–0316 Oslo, Norway

1 Introduction

The problem of a body translating on or beneath a free surface while performing an oscillating motion, is of fundamental interest in marin fluid dynamics. It is of practical importance in the seakeeping of ships, and in the studying of offshore structures and devices for exploiting wave energy. The oscillations are often of small amplitudes such that the conditions required for linearization of the problem is fulfilled. It is then appropriate to solve the problem by using a Green function. For a body moving with a constant velocity U, or equivalently, a body embedded in a uniform current $-U$, the Green function generated by an oscillating, concentrated source is well-known.

This Green function is, however, unbounded for a certain value of the frequency ω, corresponding to the non-dimensional number $\tau = U\omega/g = 1/4$. Here g is the accelaration of gravity. Physically speaking, in the two-dimensional case four waves are generated in the far-field when τ is less than $1/4$. Three of these waves have negative group velocities and are located downstream. One wave has positive group velocity and are located upstream. When $\tau \to 1/4$, two of the waves merge into one wave which has zero group velocity. This wave is not able to transport wave energy upstream and we get a wave cut-off such that the two merged waves do not exist for $\tau > 1/4$. The singularity in the Green function for $\tau = 1/4$ has therefore two causes: two of the waves merge into one which is expected to give a resonance situation, and the resulting wave is not able to transport wave energy.

The motion generated by a body of non-zero volume, oscillating or exposed to an incoming wave. may be found by using a distribution of sources located at the body surface. Since a single source is unbounded at $\tau = 1/4$, it was long believed that this is so also for a body. Grue & Palm [1] found, however, for a submerged circular cylinder in two dimensions that the motion and physical forces are bounded as $\tau \to 1/4$. The result was shown numerically as well as from the mathematical equations. Similar numerical results were obtained by Mo & Palm [2] for a submerged elliptical cylinder and by Grue, Mo & Palm [3] for a submerged foil.

This result was generalized by Liu & Yue [4] who were able to show that in two dimensions the motion at $\tau = 1/4$ is bounded for a submerged body of arbitrary form, provided that the body has a non-zero cross-section area. They also extended their theory to floating two-dimensional bodies and three-dimensional submerged bodies. The paper was followed up by a new paper, Liu & Yue [5] where their result on the motion being finite at $\tau = 1/4$, is applied to the study of the time dependence of the wave resistance of a body accelerating from rest. It is known that if the motion is started impulsively from rest to a constant translating velocity, the transient Green function decays slowly, viz. as $t^{-1/2}$ in two dimensions (and t^{-1} in three dimensions), where t is time. The reason for this slow decay is the occurrence of the singularity at the frequency corresponding to $\tau = 1/4$. It was shown that for bodies with non-zero volumes the transient motion decays an order faster: as $t^{-3/2}$ in two dimensions (and t^{-2} in three dimensions). For bodies of zero volumes they find that the decay is the same order as for the single source, however.

There are still shortcomings with the mathematical description of the physical problem at τ close to 1/4. The first relates specifically to the work [4] in which it is claimed that a finite solution exists if and only if the cross-section area is non-zero. We prove here that a finite solution of the problem exists for the motion near the singularity also when the body has zero cross-section area, namely for a thin two-dimensional foil. The result is independent of the value of the velocity circulation around the foil. Secondly, for a body with finite submergence the mathematical solution of the physical problem is bounded for $\tau = 1/4$. This solution tends, however, to infinity as the submergence of the body tends to infinity, as noted by Zhang & Zhu [6]. Such a behaviour is of course meaningless from a physical point of view.

2 Mathematical Formulation

a. Bodies of non-zero cross-section

We consider a body in two dimensions embedded in a uniform current beneath a free surface, performing small oscillations in heave or sway. There may in general also be an incoming wave of the same frequency. It is assumed that a velocity potential φ exists, satisfying the Laplace equation. φ is properly divided into two parts: $\varphi = \varphi_0 + \varphi_1$ where φ_0 is the potential of the incoming wave. φ_0 and φ_1 may be written

$$\varphi_0 = \mathrm{Re}_j \mathrm{Re}_i f_0(z) \exp(j\omega t), \quad \varphi_1 = \mathrm{Re}_j \mathrm{Re}_i f_1(z) \exp(j\omega t) \tag{2.1}$$

where $f_0(z)$ and $f_1(z)$ are analytic functions of $z = x + iy$ with x and y being the horizontal and vertical coordinates, respectively. Origin is in the undisturbed free surface and y is positive upwards. $f_1(z)$ is written as

$$f_1(z) = \int_S \sigma(s) G_\sigma(z, \zeta(s)) ds \tag{2.2}$$

where $G_\sigma(z, z_0)$ is the Green function for the problem (concentrated source at $z = z_0$). The contour of the body is determined by $z = \zeta(s)$ where s is the arclength, and $\sigma(s)$ is the source strength. The integral equation for σ is singular for $\tau = 1/4$. Near this singularity the integral equation takes the form

$$\sigma(s') + \frac{2k}{\delta}[(n_x(s') + jn_y(s')) \exp(-jk\zeta(s'))] \int_S \sigma(s) \exp(jk\bar{\zeta}(s)$$
$$+ \int_S \sigma(s) M(s, s') ds + O(\delta) = H(s') \tag{2.3}$$

Here

$$\delta = (1 - 4\tau)^{1/2}, \qquad k = \omega/U \,,$$

n_x and n_y are the x- and y-components of the normal vector of the body, and M and H are non-singular functions. A bar indicates the complex conjugate.

In [4] (2.3) was transformed into the following form

$$\sigma(s') - \frac{2k(n_x + in_y) \exp(-jk\zeta(s'))}{\delta + 2jk\Gamma} \int_S \sigma(s) ds \int_S M(s, s') \exp(jk\bar{\zeta}(s') ds'$$

$$+\int_S \sigma(s)M(s,s')ds = H(s') - \frac{(n_x + in_y)\exp(-jk\zeta(s')}{\delta + 2jk\Gamma}\int_S H(s')\exp(jk\overline{\zeta}(s')ds' \quad (2.4)$$

where

$$\Gamma = 2k\int_B \exp(2ky)dB \quad (2.5)$$

and B denotes the body section. Since $\Gamma \neq 0$, all the terms in (2.4) are finite and hence σ is finite also for $\delta = 0$.

b. Bodies of zero cross section (the foil)

We consider now a thin oscillating foil submerged in a uniform current under a free surface [3]. A thin moving foil may be used to extract wave energy. It is assumed that the foil has a small camber and angle of attack. For the oscillatory part of the flow the effects of camber and thickness are only secondary and the foil may mathematically be replaced by a flat plate. The boundary conditions at the free surface may be linearized, even if the foil is placed close to the free surface.

The velocity circulation around the foil oscillates in time. Hence vortices are shed at the trailing edge and an infinite long vortex wake will be formed behind the foil. $f_1(z)$ may now be written

$$f_1(z) = \int_{-\infty}^{\ell} \gamma(\xi)G_\gamma(z,\xi - id)d\xi \quad (2.6)$$

where $G_\gamma(z, z_0)$ is the Green function for the problem (concentrated vortex at $z = z_0$) and d is the depth of the foil. γ is now given by an integral equation of the form

$$\gamma(x) = \frac{1}{\pi^2}(\ell^2 - x^2)^{-1/2}\left[-\frac{AR}{\delta}\fint_{-\ell}^{\ell}\frac{(\ell^2-\eta^2)^{1/2}}{x-\eta}\exp(-jk\eta)d\eta + \right.$$

$$\left.\fint_{-\ell}^{\ell}\frac{\ell^2-\eta^2}{x-\eta}\left(-\int_{-\ell}^{\ell}\gamma(\xi)K_1(\eta,\xi)d\xi + H(\eta) + F_0(\eta) - \widehat{F}(\eta)\right)d\eta - \pi\Gamma_s\right] \quad (2.7)$$

Here

$$\frac{R}{\delta} = \frac{T_1 + \Gamma_s[J_0(k_1\ell) - k\exp(-jk_1\ell)/(k+k_1)]}{\delta + jg_1(k_1\ell)} \quad (2.8)$$

with k_1 denoting the wave number for one of the two waves which merge at $\tau = 1/4$. Γ_s is the velocity circulation and J_0 denotes Bessel function of the first kind of order zero. It is proved that $g_1(k_1\ell)$ is always positive. The other functions involved are all regular.

Discussion

It is noted that (2.7) and (2.8) define an integral equation where all terms are finite for $\delta \to 0$, even though the cross-section is zero. It is easily shown that also $f_1(z)$ is finite at this limit. The result is true independent of the value of the velocity circulation.

It should also be noted that the equations (2.4) and (2.7) are singular at $\delta \to 0$, even though σ and γ are finite at this limit. Physically this may be explained by the fact that for $\tau < 1/4$ four waves are present, while two of these disappear for $\tau > 1/4$. It will be shown that some of the physical forces may have infinite derivatives with respect to ω at $\tau = 1/4$. This is also true for the source strengths σ and γ.

It is seen from (2.4) that for $\delta = 0$ the second term becomes for large submergence proportional to $\exp(kd)$ (d the depth). Also the velocities become proportional to $\exp(kd)$, which is a meaningless result from a physical point of view. A similar behviour is also found for the foil. It was proposed in [6] to solve the problem by introducing non-linear effects. They exploit a quasi-linear model using a Green function, originally derived by Dagan & Miloh [7], which satisfies the free surface conditions up to third order in the small parameter ϵ. By this they obtain that the wave motion set up at the free surface by a deeply situated body *decays* exponentially with the submergence of the body at $\tau = 1/4$ which is a reasonable result. However, we believe that any other Green fuction may be used, having the merit that for $\epsilon \to 0$ the classical Green function is recovered, and for deeply submerged bodies the wave motion at the free surface decays rapidly with the depth of the body.

We conclude that there are three different routes which may be taken to solve the problem. The first one is to integrate the integral equation directly through the singularity ([1], [2], [3]) which can be performed without difficulty. The second one is to develop the integral equation to a form where the source strength is finite at $\tau = 1/4$ [4], and the third one is to use a Green function which is non-singular at $\tau = 1/4$ [6]. Each of these methods have their advantages and disadvantages.

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