Prediction of resonances due to waves interacting with finite linear arrays of cylinders

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1 Introduction

In this abstract we show how information concerning the trapped modes in the vicinity of an infinite array of bottom-mounted cylinders can be used to make accurate predictions of the frequencies at which large forces will occur on finite arrays of cylinders. Results are given here for circular cylinders, and it is hoped further results will be presented at the Workshop. Recently Maniar & Newman (1997) have shown how the interaction between an incident wave field and a long periodic array of vertical circular cylinders extending throughout the depth can generate large free-surface amplitudes and forces on the cylinders. They found that the frequencies at which these large resonances occurred corresponded to frequencies at which trapped modes exist for the corresponding *infinite* array of cylinders. Trapped modes represent a localised oscillation of finite energy which does not propagate away to infinity and they are simply the non-trivial solutions to the homogeneous problem. Using symmetry arguments in the trapped mode problem, the infinite array can be regarded as being equal to the problem of a single cylinder placed symmetrically in a channel with parallel walls having either Neumann or Dirichlet condition imposed upon them. Furthermore, it is also necessary to place a Dirichlet (antisymmetry) condition on the channel centreplane in order to generate a cut-off frequency. For the channel with Neumann conditions on the walls, this cut-off is given by $kd = \frac{1}{2}\pi$ where k is the wavenumber and the channel is of width $2d$. For the channel having Dirichlet conditions on the walls, the cut-off is at $kd = \pi$. In each case, provided that the wavenumber is below its respective cut-off and provided that the motion is antisymmetric about the channel centreline, any oscillation localised about the cylinder is unable to propagate to infinity along the channel and is therefore trapped. The Neumann trapped mode was first shown to exist for circular cylinders of all sizes $0 < a/d \leq 1$, with a the cylinder radius, by Callan *et al* (1991). The Dirichlet trapped modes computed by Maniar & Newman (1997) only exist if $0 < a/d < 0.678$, that is for sufficiently small cylinders. Evans *et al* (1994) proved that all symmetric obstacles placed symmetrically in a channel having Neumann conditions on the walls exhibit a trapped mode below the cut-off $kd = \frac{1}{2}\pi$. The same is not true for a channel having Dirichlet conditions on the walls (as demonstrated, for example, by the circular cylinder), though the techniques used in Evans *et al* (1994) can be adapted to the Dirichlet case to provide a powerful result for the existence of Dirichlet trapped modes. More recently, Evans & Porter (1997) have shown that further isolated trapped modes exist *above* the cut-off for the circular cylinder in both the Neumann and Dirichlet case. In each case, they only exist at a precise wavenumber and for a precise cylinder size.

All the resonances appearing for a finite periodic array of cylinders in waves can be attributed to the presence of one of these trapped modes for the infinite array (see figure 1(a)). However, the values of kd at which maximum response occurs for the finite array is dependent on the number of elements, N, in the array and only tends to the trapped mode wavenumber as $N \to \infty$. Similarly, the amplitude of resonance increases (roughly linearly with N) as N increases, though for an infinite array the response would be infinite. In the present paper we attempt to go further by predicting the value of kd and the amplitude of resonance for a finite array of N elements using only information from an infinite array. Though this appears on the face of it to be a step backwards, an infinite array

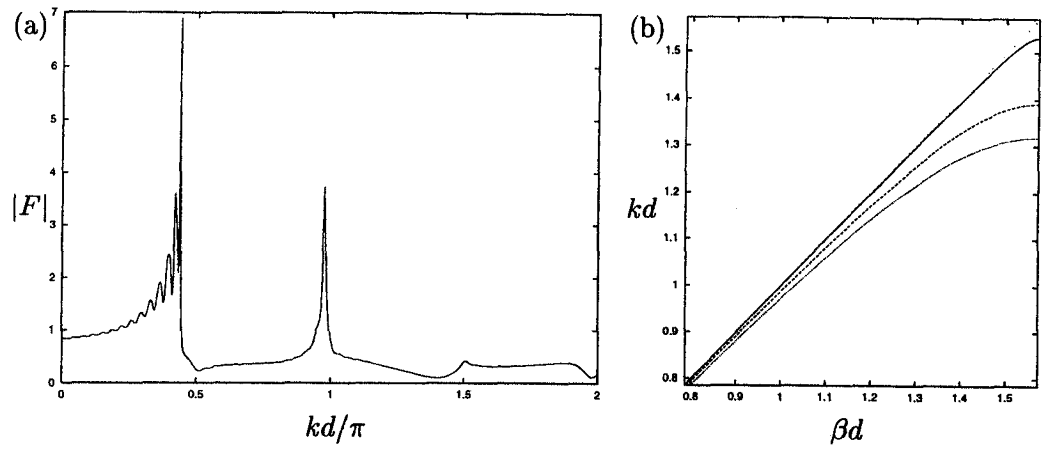

Figure 1: (a) Maximum exciting force on the middle cylinder in array of 25 cylinders in head seas with $a/d = \frac{1}{2}$. (b) kd versus βd for Rayleigh-Bloch waves along an array of circular cylinders with $a/d = \frac{1}{4}$ (—), $\frac{1}{2}$ (- - -), $\frac{3}{4}$ (···).

is simpler to deal with analytically and so this concept provides a useful tool for predicting forces and frequencies on finite arrays of cylinder with general cross sections.

The trapped modes described above are just special cases of a more generalised trapped mode motion which are usually referred to as Rayleigh-Bloch waves (sometimes also called guided waves or edge waves).

Briefly, Rayleigh-Bloch waves describe oscillations in the vicinity of a periodic array or grating which do not radiate energy away from the grating but, in general, have some transport of energy along the array. They are characterised by a dominant wavenumber β in the direction of periodicity, the wavenumber k then having to satisfy the cut-off criterion $k < \beta$ so as to ensure no outgoing waves. Thus $\beta d = \frac{1}{2}\pi$ and $\beta d = \pi$ are equivalent to the Neumann and Dirichlet trapped modes described earlier. Rayleigh-Bloch waves are explained in more detail in the following section.

2 Rayleigh-Bloch waves along periodic gratings

Consider an infinite periodic linear array of cylinders each of arbitrary cross section, having boundary ∂D, uniform throughout the depth. The generators of the cylinders are aligned with the depth coordinate, z, and positioned at $(x, y) = (0, 2jd)$, where j is an integer running from $-\infty$ to ∞. According to classical linearised theory and assuming time harmonic motion whilst also removing the depth variation through a term proportional to $\cosh k(z-h)$ where h is the constant fluid depth, the two-dimensional complex velocity potential describing the flow satisfies the Helmholtz equation,

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0 \tag{1}$$

everywhere in the field apart from on the boundaries of the cylinders where

$$\phi_n = 0, \tag{2}$$

and n denotes the normal derivative with respect to the cylinder surface. Because the geometry has periodicity of $2d$ in the y-direction, we may relate the potential through

$$\phi(x, y + 2jd) = \mathrm{e}^{\mathrm{i}\beta 2dj}\phi(x, y), \qquad -\infty < j < \infty \tag{3}$$

which simply expresses that there is a change in phase of $\mathrm{e}^{\mathrm{i}\beta 2d}$ from the field point at y to the field point at $y+2d$ in the adjacent 'cell'. Thus the total field can be obtained by referring to a single strip of width $2d$ containing the cylinder. We therefore restrict our attention to the strip $(x, y) \in (-\infty, \infty) \times [-d, d]$ and impose appropriate periodicity conditions on the lines $y = \pm d$ of

$$\phi(x, d) = \mathrm{e}^{\mathrm{i}\beta 2d}\phi(x, -d), \qquad \phi_y(x, d) = \mathrm{e}^{\mathrm{i}\beta 2d}\phi_y(x, -d), \tag{4}$$

with (3) providing the extension to all (x, y). The Green's function for the problem defined by (1), (2) with (4) may be written as the integral representation

$$G_\beta(x,y;\xi,\eta) = -\frac{1}{2\pi} \int_0^\infty \frac{e^{i\beta d \operatorname{sgn}(y-\eta)} \sinh k\gamma|y-\eta| + \sinh k\gamma(d-|y-\eta|)}{\gamma(\cosh k\gamma d - \cos\beta d)} \cos k(x-\xi)t \, dt \tag{5}$$

where $\gamma = (1-t^2)^{1/2} = i(t^2-1)^{1/2}$ and $r = ((x-\xi)^2-(y-\eta)^2)^{1/2}$. See Linton (1998) for its derivation and other representations of the periodic Green's functions in (5). Applying Greens theorem to G_β and ϕ in the rectangle $(x, y) = (-X, X) \times [-d, d]$, $X \to \infty$ yields the following integral equation for ϕ:

$$\int_{\partial D} \phi(p) \frac{\partial}{\partial n_q} G_\beta(p;q) \, ds_q = \begin{cases} \frac{1}{2}\phi(p), & p \in \partial D, \\ \phi(p), & p \notin \partial D. \end{cases} \tag{6}$$

Following Linton & Evans (1992), we use a polar parametrisation for ∂D of $\rho(\theta)$, $0 \le \theta \le 2\pi$ and write (θ, ψ) for (p, q). Discretising the integral equation into M segments over the interval $(0, 2\pi)$, writing $\theta_j = (2j-1)\pi/M$, $j = 1, \ldots, M$ and collocating reduces the above to the following algebraic system of equations:

$$\frac{2\pi}{M} \sum_{j=1}^{M} \phi(\theta_j) K_{ij} w_j = \tfrac{1}{2}\phi(\theta_i), \qquad i = 1, \ldots, M \tag{7}$$

where

$$K_{ij} = \partial G_\beta(\theta_i;\theta_j)/\partial n_q, \qquad w_j = (\rho^2(\theta_j) + \rho'^2(\theta_j))^{1/2}. \tag{8}$$

It turns out that if we are below the cut-off, $k < \beta$, the above system can be recast as a real system despite the apparent complex nature of G_β in (5). Rayleigh-Bloch modes correspond to the non-trivial solutions to (7) or, equivalently, the vanishing of the determinant of the system, for which the realness of the system is vital.

For example, when the infinite array consists of circular cylinders of radius a, the Rayleigh-Bloch solutions in $\beta d \le \frac{1}{2}$ are shown in figure 1(b). Notice that $\beta d = \frac{1}{2}\pi$ corresponds to a Neumann trapped mode, whence the well-known results of Callan *et al* (1991) are recovered.

3 Near-trapping by a finite linear array

In figure 1(a) we show the variation of the maximum exciting force on the middle cylinder in an array of 25 cylinders of non-dimensional radius $a/d = \frac{1}{2}$ with non-dimensional wavenumber kd/π. We are interested in predicting the values of kd at which large peaks in forces occur. Maniar & Newmann (1997) made the connection between the Neumann and Dirichlet trapped modes in an infinite array ($\beta d = \frac{1}{2}\pi$ and $\beta d = \pi$ respectively) and these peaks. In fact, for 25 cylinders, the peak resonance occurs at a value of $kd = 1.3820$ as opposed to the corresponding Neumann trapped mode wavenumber of $kd = 1.3913$. In what follows, we allow ourselves to consider general βd and the resulting Rayleigh-Bloch waves in the infinite array to improve upon the estimate to kd at which resonance occurs for a finite array. Our motivation comes from the form of the wave field along the finite array, shown for 25 cylinders at the resonant wavenumber $kd = 1.3820$ in figure 2(b). In each 'cell' containing a cylinder, the wave field is similar to that for a trapped mode in a Neumann channel, but is modulated by a cosine-type envelope along the array. We can construct a similar solution for the Rayleigh-Bloch waves by choosing $\beta d = \frac{1}{2}\pi(1-\varepsilon)$. Then from (3),

$$\phi(x, y + 2jd) = e^{ij\pi(1-\varepsilon)}\phi(x,y) = e^{-ij\varepsilon}[(-1)^j \phi(x,y)]. \tag{9}$$

The term in the square brackets represents the standing wave component of the solution whilst the exponential term contains a modulation of one half wavelength given by $j\varepsilon = 1$. Matching this modulation with the finite array of N elements gives $N\varepsilon = 1$, and so

$$\beta d = \tfrac{1}{2}\pi(1 - 1/N) \tag{10}$$

and the corresponding Rayleigh-Bloch wavenumber $kd(\beta d)$ can be computed using the method outlined in the preceding section. This resulting value of kd provides the estimate to the wavenumber at which

N (no. of cylinders)	$\beta d = \frac{1}{2}\pi(1-1/N)$	$kd(\beta d)$	kd(peak force)
100	1.5550	1.3907	1.3907
50	1.5394	1.3889	1.3889
25	1.5080	1.3818	1.3820
20	1.4923	1.3767	1.3775
15	1.4661	1.3659	1.3680
10	1.4137	1.3376	1.3470

Table 1: Table showing the values of kd at which large forces occur in a linear array of N cylinders, $a/d = \frac{1}{2}$ and the wavenumbers predicted using Rayleigh-Bloch theory.

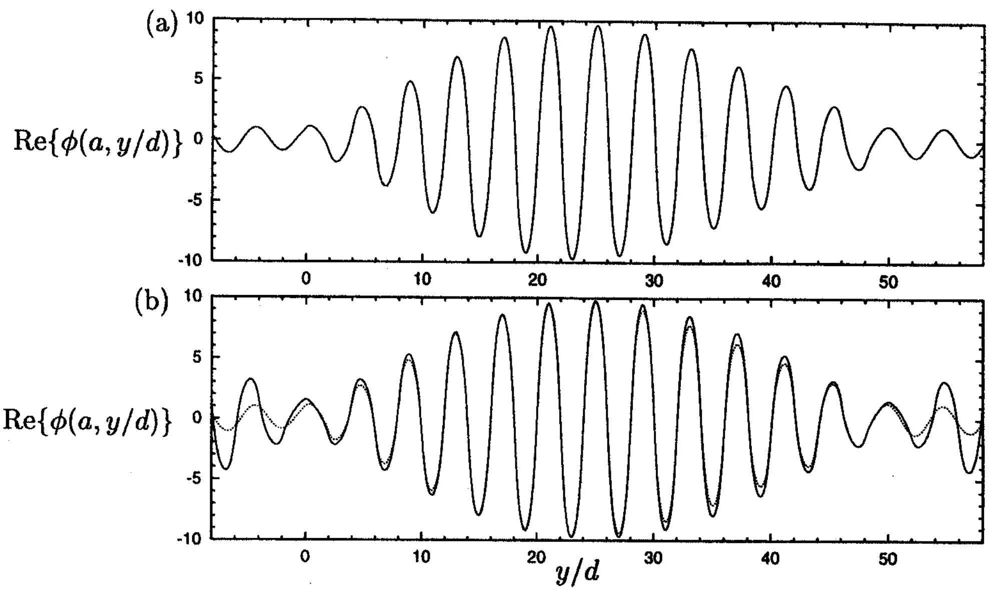

Figure 2: (a) The free-surface elevation along $N = 25$ cylinders at the near-trapping frequency ($a/d = \frac{1}{2}$), and (b) overlayed ($\cdots$) on the Rayleigh-Bloch surface profile (—) along a corresponding infinite array with βd given by (10).

the peak resonance occurs in the finite array and a comparison between the two is shown in table 1. For $N \geq 25$ the agreement is excellent and even for $N = 10$, the discrepancy is only 1%. Figure 2(b) shows an overlay of the wave profile along the finite array of $N = 25$ cylinders and the corresponding Rayleigh-Bloch wave profile computed using (10). In the range occupied by cylinders, the agreement is excellent, confirming the connection between near-trapping or resonance in finite linear arrays and Rayleigh-Bloch waves in infinite arrays.

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