A Waterfall Springing from Unsteady Flow over an Uneven Bottom

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1. Introduction

The goal of the research presented here is to investigate the unsteady flow over an uneven bottom resulting in a waterfall at its terminus as shown in Figure 1. This is a problem that has not received much attention either through theoretical investigation or through experimentation. Such a flow might arise from a stream with waves approaching the waterfall or even from an earthquake causing an uplift of the stream bottom near the waterfall. This flow itself is clearly not of immediate practical significance, especially for the naval architect or ocean engineer, but it is related in a way to the flow in a plunging breaking wave as we shall describe later in this paper. The work described here is still in its formative stage and the results presented here are only for the steady flow over an uneven bottom.

2. Model

The model chosen for this study, the Green-Naghdi (GN) method of fluid sheets, always yields a three-dimensional, unsteady model for such flows. The GN approach is a continuum model in which the kinematic character of the flow is prescribed in the vertical direction. With this restriction, the equations for modeling the flow satisfy the boundary conditions exactly, satisfy conservation of mass and momentum exactly and are Gallilean invariant. In the GN method surface tension and viscosity can be included without real penalty although such flows are limited to laminar flows (see, for instance, Kim and Webster, 1995). For the flow here, both surface tension and viscosity will be neglected. Different levels of GN theory depending on the degree of specified kinematic complexity in the flow.

This approach is very different from the more classical approach in that the model for the flow (inviscid flow) is combined with the simplification (proscription of the vertical kinematic complexity) right from the outset. In the classical approach, the model of inviscid flow as a potential flow field is developed with an appeal to Kelvin's theorem. Subsequently a simplification, usually perturbation scheme involving a systematic expansion of the field equations and boundary conditions, is involved using a small parameter as a gauge for retaining or discarding terms.

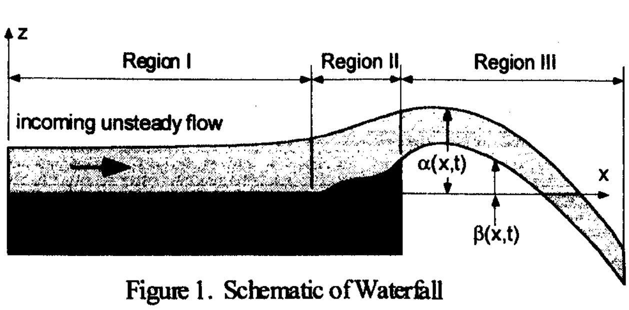

Figure 1. Schematic of Waterfall

In the end, each approach has its advantages and its blemishes. In the GN approach, the boundary conditions and the conservation laws are satisfied exactly but the fluid field is not exactly irrotational. In the classical approach, the fluid field is irrotational but the boundary conditions and the conservation laws are only approximately satisfied (i.e., satisfied only up to the order of the terms discarded in the expansions). Another way of looking at the difference is that the classical method is correct locally but approximate globally, and the GN method is the opposite. As with all modeling problems, determination of which approximation scheme is the most appropriate for a given problem must be left to comparison with physical experiments. After all, both approaches ignore the vorticity due to viscosity that is certainly there. Some approximation schemes result in models with other blemishes. For instance the Korteweg-DeVries (KdV) and super KdV models are not Gallilean invariant.

3. Waterfall problem

Consider the steady two-dimensional flow of an incompressible, inviscid fluid under the action of gravity over a cliff leading to a free overfall (as shown in Fig. 1). Three distinct regions of flow may be associated with this problem. The upstream region (region I) is characterized by a free top surface and an even bottom. The middle region (Region II) is characterized by a free top surface and an uneven bottom. For the study here we shall restrict the unevenness to be a uniform slope, although there is no limitation in the theory in this regard. In the downstream region (labeled as III) both the top and bottom surface of the fluid are free. Far upstream the fluid is assumed to flow as a uniform stream, while downstream the fluid falls freely under the action of gravity. Of particular interest in analyzing the problem is the prediction of the height of the whole flow region and the determination of the downstream solution, i.e., the shape of the free surfaces and the vertical thickness of the jet.

Green-Naghdi Theory Level-I

Here we use Green-Naghdi theory level-I with the formulae and notations derived by J. Shields and W. Webster (1988). We consider here only two-dimensional problems. The Level-I theory is summarized as follows:

The velocity profile (u,v) is assumed to be of the form:

$$u(x,z,t)=u_0(x,t)$$
$$v(x,z,t)=v_0(x,t)+v_1(x,t)\zeta. \tag{1}$$

The kinematic boundary conditions are:

$$v_o+v_1\alpha=\alpha_t+u_o\alpha_x;$$
$$v_o+v_1\beta=\beta_t+u_o\beta_x. \tag{2}$$

where $\beta=\beta(x,t)$ and $\alpha=\alpha(x,t)$ represent the top and bottom surfaces respectively.

The continuity equation is

$$u_{ox}+v_1=0 \tag{3}$$

The momentum equations are as follows:

$$u_{ot}\phi_o+\phi_o u_o u_{ox}=\frac{1}{\rho}(-P_{ox}+\hat{p}\beta_x-\overline{p}\alpha_x)$$

$$u_{ot}\phi_1+\phi_1 u_o u_{ox}=\frac{1}{\rho}(-P_{1x}+\hat{p}\beta\beta_x-\overline{p}\alpha\alpha_x)$$

$$\begin{aligned}&v_{ot}\phi_o+\phi_o u_o v_{ox}+\phi_o v_o v_1+\\&v_{1t}\phi_1+\phi_1 u_o v_{1x}+\phi_1 v_1 v_1\\&=\frac{1}{\rho}(-\rho g\phi_o-\hat{p}+\overline{p})\end{aligned} \tag{4}$$

$$\begin{aligned}&v_{ot}\phi_1+\phi_1 u_o v_{ox}+\phi_1 v_o v_1+\\&v_{1t}\phi_2+\phi_2 u_o v_{1x}+\phi_2 v_1 v_1\\&=\frac{1}{\rho}(P_0-\rho g\phi_1-\hat{p}\beta+\overline{p}\alpha)\end{aligned}$$

where $\hat{p}$ and $\overline{p}$ are the pressure on the top and bottom surfaces respectively, and where

$$\phi_o=\beta-\alpha;$$
$$\phi_1=\frac{1}{2}(\beta^2-\alpha^2); \tag{5}$$
$$\phi_2=\frac{1}{3}(\beta^3-\alpha^3);$$
$$P_n=\int_\alpha^\beta p\zeta^n d\zeta.$$

Since P_1 only exists in the second equation of (4), it is then only a dependent variable and need not be solved simultaneously with the other variables.

Formulation of the problem

A statement of the problem under consideration is given in section 1 and for this study we consider only steady flow. With reference to Figure 2, we choose the x-y co-ordinate axes as shown in Figure 2: Region I is the domain x < -a; Region II is the domain $-a \le x < 0$ and Region III is the domain $x \ge 0$. It follows that the pressure $\hat{p}$ at the top surface equals the atmospheric pressure p_0 in the whole region. The given quantities and unknowns are as follows:

$$\text{Region I } (x<-a)\begin{Bmatrix}\hat{p}=p_0,\alpha=0\\ \overline{p},\beta\ \textit{unknown},\end{Bmatrix}$$

$$\text{Region II } (-a\le x\le 0)\begin{Bmatrix}\hat{p}=p_0,\alpha(x)=Kx+Ka,\\ \overline{p},\beta\ \textit{unknown}\end{Bmatrix}$$

$$\text{Region III } (x>0)\begin{Bmatrix}\hat{p}=p_0,\overline{p}=p_0\\ \alpha,\beta\ \textit{unknown}\end{Bmatrix}$$

where K is the slope of the bottom in region II. After Simplifying, we can obtain the governing equations for three regions:

Region I:

$$\begin{cases}\frac{1}{3}Q^2\phi_{ox}{}^2-Q^2+g\phi_o{}^3-2R_1\phi_o{}^2+2S_1\phi_o=0\\ \frac{\overline{p}}{\rho}=g\phi_0-\alpha_x Q^2\frac{\phi_{0x}}{\phi_0^2}-\frac{1}{2}Q^2\frac{\phi_{0x}{}^2}{\phi_0^2}+\frac{1}{2}Q^2\frac{\phi_{0xx}}{\phi_0}\end{cases}$$

Region II: (6)

$$\begin{cases}\frac{\phi_{oxx}}{\phi_o}-\frac{1}{2}\frac{\phi_{ox}{}^2}{\phi_o{}^2}+\frac{3}{2}(1+\alpha_x^2)\frac{1}{\phi_o{}^2}\\ \qquad+\frac{3g}{Q^2}(\phi_o+\alpha)+\frac{3}{2}\frac{\alpha_{xx}}{\phi_o}=3\frac{R_2}{Q^2}\\ \frac{\overline{p}}{\rho}=g\phi_0-\alpha_x Q^2\frac{\phi_{0x}}{\phi_0^2}-\frac{1}{2}Q^2\frac{\phi_{0x}{}^2}{\phi_0^2}+\frac{1}{2}Q^2\frac{\phi_{0xx}}{\phi_0}\end{cases}$$

Region III:

$$\begin{cases}\frac{Q^2}{12}\phi_{ox}^2=Q^2-2S_3\phi_o+2R_3\phi_o^2\\ Q^2\left(\frac{\psi_x}{\phi_o}\right)_x=-g\phi_o\end{cases}$$

where R_1, R_2, R_3, S_1 and S_3 are constants of integration, which can be determined by boundary conditions and matching conditions. Q is the total flow through any section.

Boundary conditions

It is assumed in the statement of problem that far upstream the fluid flows as a uniform stream. Then the far upstream boundary conditions are as follows:

$$as\ x\to-\infty \tag{7}$$

$$\phi_o\to H_1,\ \phi_{ox}\to 0,\ u_0\to u_1,\ P_0\to\tfrac{1}{2}\rho g H_1^2$$

where the constants H_1 and u_1 denotes the depth and velocity far upstream respectively.

As for the far downstream boundary conditions, we follow Naghdi's assumption, i.e., far downstream the pressure distribution (in the three-dimensional theory) is uniform throughout the thickness of the fluid sheet and is equal to the atmospheric pressure p_0. This assumption leads to the following boundary conditions:

$$as\ x \to +\infty \qquad (8)$$
$$\phi_o \to H_4,\ \phi_{ox} \to 0,\ \phi_{oxx} \to 0,\ P_0 \to 0$$

where the constant vertical thickness H_4 of the fluid sheet far downstream is to be determined in the course of solution.

Matching conditions

In order to obtain a solution which holds throughout ($-\infty < x < +\infty$), the solutions in region I, II and III must be matched at x=-a and x=0. This matching is accomplished by using the standard jump conditions associated with the integral balance laws of the theory of a directed fluid sheet. Assuming that the fluid flows smoothly at x=-a and leaves the edge of the cliff smoothly at x=0, the appropriate two-dimensional form of the jump conditions for a fluid sheet of variable initial depth may be written as:

$$[u_o\phi_o]\|_{x=-a} = 0; \quad [\phi_o]\|_{x=-a} = 0;$$
$$[\phi_{ox}]\|_{x=-a} = -K; \quad [P_o]\|_{x=-a} = 0 \qquad (9)$$

and

$$[u_o\phi_o]\|_{x=0} = 0; \quad [\phi_o]\|_{x=0} = 0;$$
$$[\phi_{ox}]\|_{x=0} = 0; \quad [P_o]\|_{x=0} = 0$$

where the notation $[f]$ stands for

$$[f]\|_x = f^{x+} - f^{x-}.$$

Results

Unfortunately, to date we have not been able to find experimental data for comparison with this development. Naghdi and Rubin (1981) using different (but equivalent) form of the Green-Naghdi method analyzed the waterfall springing from the flow over a flat bottom. For this case there are some experimental results from Rouse (1936). Figure 2 shows our calculated profile for this special case for Fr = 2.0, and H_1 = 0.9201 meter. The shape and particularly the value of H_3 = 0.8889 meter (at x = 0) agree extremely well with the experiments.

Several cases with sloped bottoms have been calculated. The upstream height H_1 = 1 meter. Figure 3 shows the flow profiles of fluid sheet with different upstream Froude Number (Fr=1.25, 1.5, 2.5 and even 8.0) for a bottom slope K = 0.1.

4. Future Research

Progress is now being made on performing similar calculations using GN Level II theory where the kinematic model for the vertical variation in velocity involves an additional term in both the horizontal and vertical velocities. That is, for Level II, the kinematic approximation corresponding to (1) is

$$u(x,z,t) = u_0(x,t) + u_1(x,t)\zeta$$
$$v(x,z,t) = v_0(x,t) + v_1(x,t)\zeta + v_2(x,t)\zeta^2 \qquad (10)$$

With this kinematic model it is possible to treat the flow over a weir (Figure 4). In particular, it is possible to model the jet streaming vertically from the gate of the weir.

Relation to Plunging Breakers

Consider the flow in a wave approaching a beach and experiencing the effects of shoaling. At any instant of its evolution, a stagnation streamline separates the flow downstream of the crest (i.e., towards the beach) from that upstream (towards the ocean). Before the wave breaks this streamline terminates at the crest. However, as the wave begins to break, this streamline bends over and a jet is formed creating the plunging part of the breaker. The upstream flow including the jet is not unlike the unsteady flow over a weir as sketched in Figure 5. Use of the GN method to model this evolution would require treating the downstream (beachside) flow as a separate fluid sheet with appropriate matching conditions. Further, such a model would also require some treatment of the impact of the jet with the water in the downstream sheet. This process is clearly non-conservative and would take some care to develop.

5. References

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6. Acknowledgement

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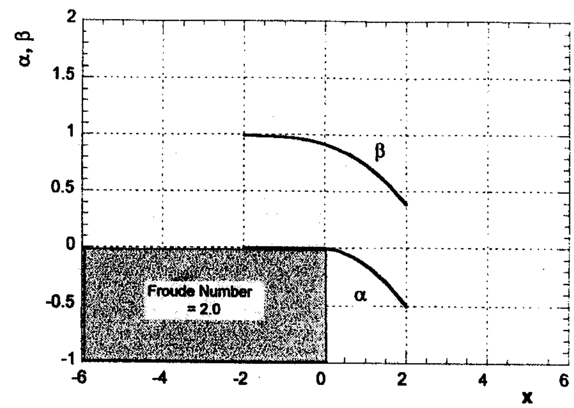

Figure 2. Waterflow springing from a flat bottom

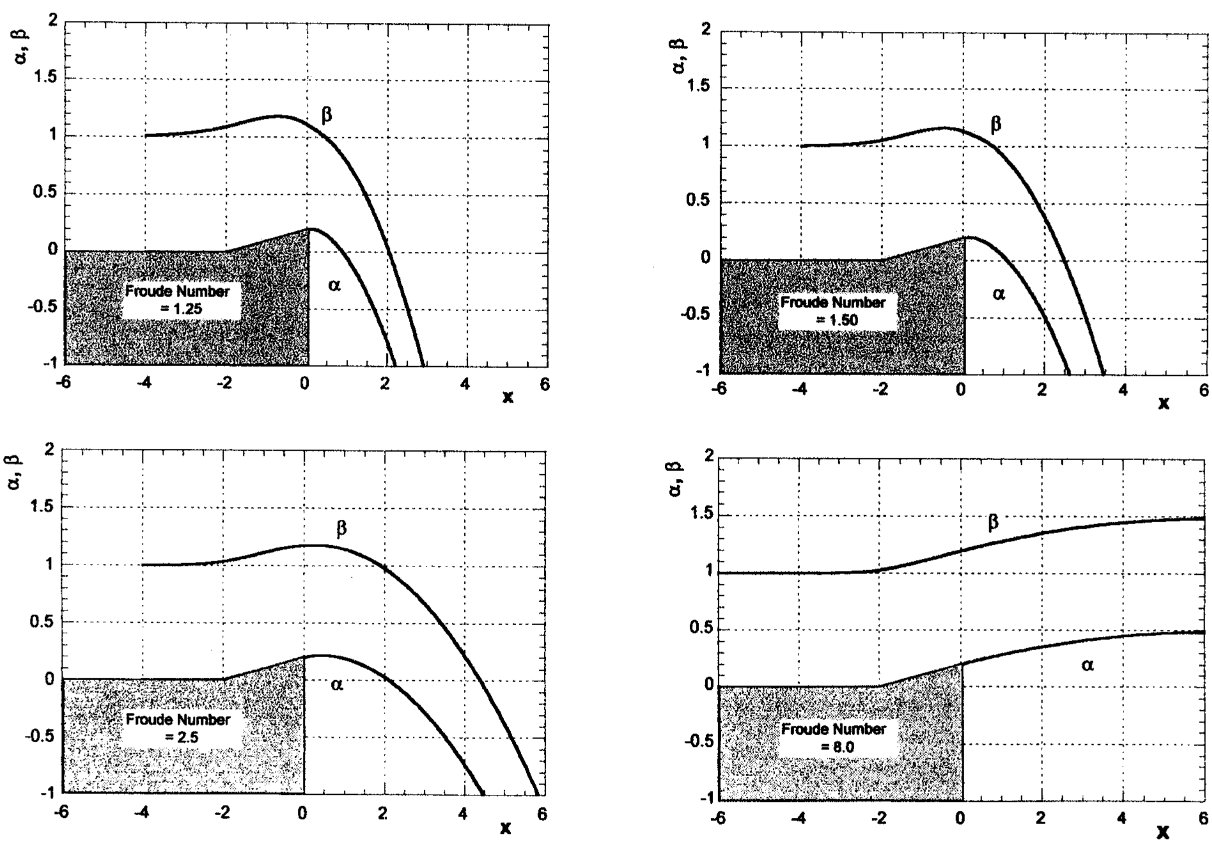

Figure 3. Effect of Froude number on waterfall springing from a sloped bottom

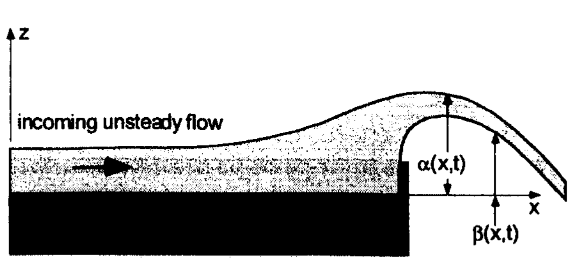

Figure 4. Schematic of Flow over a Weir

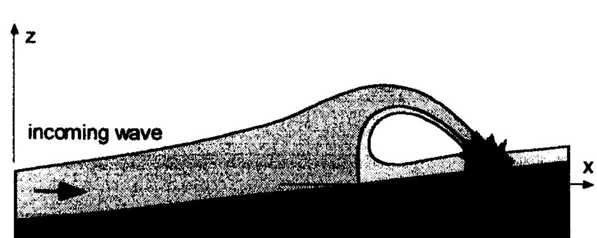

Figure 5. Schematic of Plunging Breaker Flow