Applying the Finite Element Method in numerically solving the two dimensional free-surface water wave equations

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1 Introduction

As part of the research project 'a scientific infrastructure for laboratory generated surface waves', we have been developing a computer code to simulate free surface water waves on a two dimensional bounded domain. It is well known that if the water is assumed to be inviscid, incompressible and irrotational, the velocity field of the water can be characterized with a potential function.

The set of equations describing the dynamic behaviour contains two time dependent conditions at the free surface (dynamic and kinematic boundary conditions) and Laplace's equation for the potential in the interior of the water domain. At this moment we have assumed the other boundaries to be fixed and impermeable, but our research aims at including moving wave generators and beaches.

When the equations are linearized, they can be solved in the frequency domain, but for some applications solutions of the original equations are necessary. In order to solve the nonlinear equations, the nonlinear time dependent free surface equations have to be integrated over time and at every time stage Laplace's equation has to be solved on the region bounded by the free surface and the fixed walls.

Solving Laplace's equation is the most computer-time consuming part of the numerical computations. For this reason a boundary integral description of Laplace's equation is usually discretized (e.g. boundary element method), thus reducing the number of unknowns. However, computing the coefficient matrix and solving the full matrix associated with the BEM formulation are computational intensive procedures.

Instead of using a boundary element method, we have implemented a finite element method (triangular elements and linear base functions) to solve Laplace's equation. The use of a finite element method was initiated by the article 'A finite element method for fully nonlinear water waves' by Xing Cai, et. al. (1996). In this article a method based on a time-dependent mapping of the water domain to a fixed computational rectangle is proposed. Numerical calculations have shown however that discretizing the domain directly and thus regridding the nodes at every time-stage gives more accurate results.

Although the number of unknowns using FEM is larger than using BEM, evaluation of the elements of the associated sparse matrix is relatively fast and because of the banded and symmetric structure of the matrix an efficient Gauss-Elimination solver can be used.

2 Contents of the presentation

We will discuss the benefits and limitations of applying a finite element method to solve the nonlinear wave equations numerically. The main advantages seem to be:

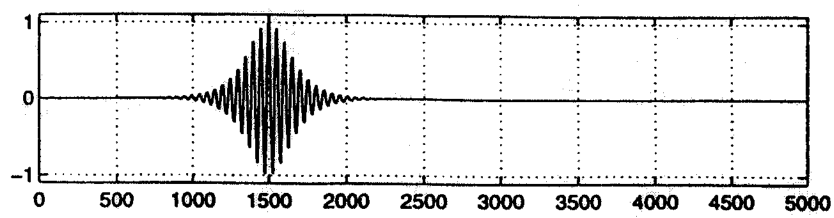
- speed and memory usage: we have applied our code (on a pentium PC) to a wavegroup propagation problem that could not be computed using a BEM without domain decomposition (on a Cray C98)
- flexibility: the finite element grid is constructed inside the domain, providing more control over the numerical accuracy near critical geometries.

Results will be presented of comparisons in which we have applied the numerical code to the following three problems that have relative simple geometries:

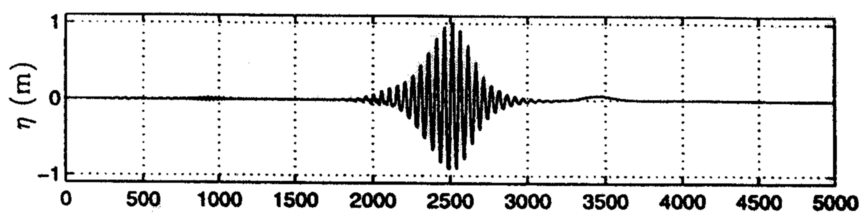
- Sloshing wave: compared with results of the sloshing wave problem in 'Comparative study of fully non-linear wave simulation programs' initiated by Det Norske Veritas, 1994. Given the dimensions of the water tank ($70m \times 160m$) and an initially steady surface profile, participants in the comparative study were asked to compute the surface elevation at t=9.2s at x=60m. For our computations we used a 70×160 grid, a 5 stage 4'th order RK method and a timestep of $\Delta t = 0.1$. The table below summarizes their results added with the result obtained by using our code.

part. nr.	resuls	part nr.	result
1	-3.803	5	-3.820
2	-3.860	6	-3.803
3	-3.815	7	-3.720
4	-3.759	our result	-3.798

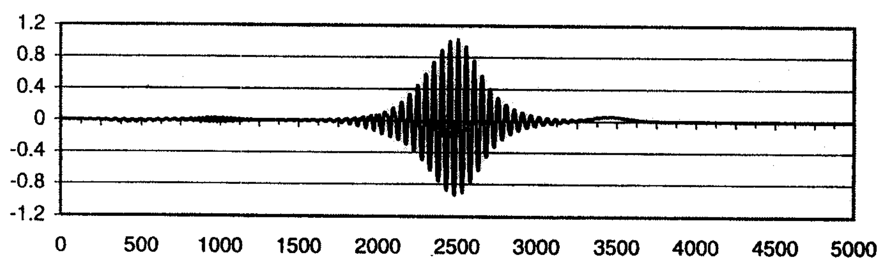
- Propagation of wavegroups: compared with results in the PhD thesis 'Numerical simulation of nonlinear water waves using a panel method; domain decomposition and applications' by Paul de Haas, 1997. The figures on the next page show the initial surface of the wavegroup (propagating to the right), the result obtained by Paul de Haas at $t = 180$ and the result of our computations at $t = 180$ using a 2001×7 grid. The depth of the water is 12 meters and in both computations $\Delta t = 0.3$



- Wavegroup: initial surface profile of the wavegroup

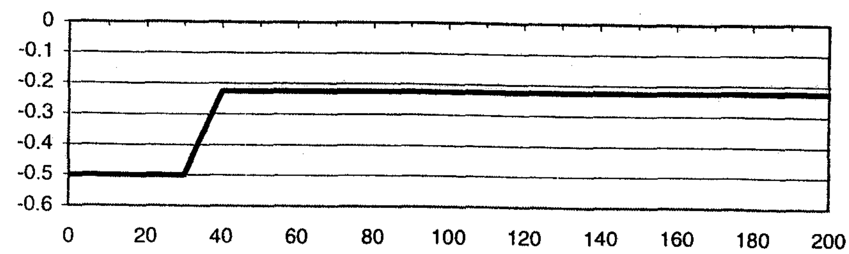


- Wavegroup: surface after 180 seconds computed by Paul de Haas using a panel method and domain decomposition (copied from his thesis)

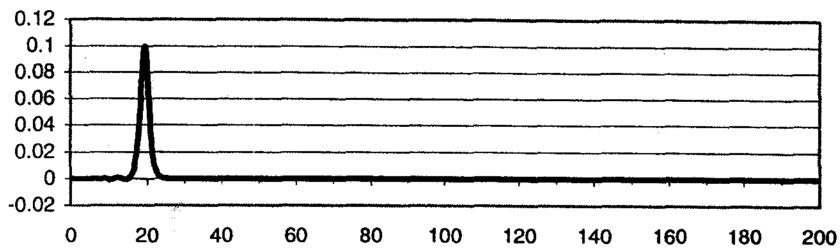


- Wavegroup: surface after 180 seconds computed with our code (no domain decomposition and on a desktop PC)

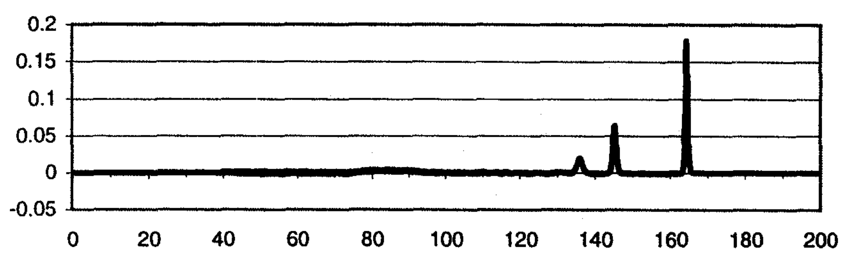
- Soliton splitting over a varying bottom: compared with results from the paper 'BEM-numerics and KdV-model analysis for solitary wave split-up' by E. van Daalen, E. van Groesen and S. Pudjaprasetya in Computation Mechanics, vol 19: 197-187 ,1997. The figures on the next page show the topography and the computed surface at t=8 and t=80 of a solitary wave propagating to the right and splitting into three solitons. For this computation a uniform grid (2001 × 6) and a 5 stage 4'th order Runga-Kutta time integration with $\Delta t = 0.1$ were used.



-Soliton splitting: used bottom topography



-Soliton splitting: solitary wave before splitting at $t = 8s$ propagating to the right.



-Soliton splitting: $t = 80s$, original solitary wave has split into three separate solitons

3 Conclusion

A FEM based numerical solver for the kind of problems as described above seems to be a good alternative for conventional boundary integral methods. Although more work has to be done to investigate accuracy, stability and applicability to a wider range of problems, results so far are encouraging. Future objectives are to implement higher order FEM base functions, incorporate moving boundaries, introduce realistic wave absorbers (as are being used in hydrodynamic laboratories) and to implement the code for three dimensional situations.