Stability of time-domain Boundary Element Models; Theory and applications

Bjarne Büchmann[1] and Jesper Skourup[2]

Motivation

At previous workshops some attention has been given to the stability analysis of time domain Boundary Element Models (BEMs) for the interaction between waves and a floating body (see *e.g.* Vada and Nakos, 1993; and Bunnik and Hermans, 1998). Instability problems have often been observed in these models, so in order to generate numerically stable models, the causes for these instabilities must be investigated.

Since time domain simulation of three-dimensional fully non-linear hydrodynamic free-surface problems still tends to be very demanding with respect to computing time (see *e.g.* Celebi *et al.*, 1998), perturbation based finite order time-domain models are often utilized (see *e.g.* Büchmann *et al.*, 1998). However, even for models where a thorough stability analysis has been made and the stability criterion is not violated, filtering may still be needed to avoid short wave instabilities (see *e.g.* Kim *et al.*, 1997). This apparent discrepancy between the predictions of the stability analyses and the observations from the time domain models motivates this abstract. It is the hope of the authors, that this abstract and the following presentation will inspire to a lively and fruitful discussion on the (in)stability of time-domain BEMs, the reasons for instabilities as well as methods to avoid them.

Numerical instabilities in BEMs

Since the first publications where BEMs were used for modeling of free surface waves in the time domain (see *e.g.* Longuet-Higgins and Cokelet, 1976) short wave instabilities (the so-called "wiggles") have been observed, and various techniques have been utilized to remedy the problem. Dommermuth and Yue (1987) showed that control over the minimum grid size is necessary for stability of the fully non-linear Eulerian–Lagrangian time domain BEMs. Thus, "regridding" techniques are often utilized to stabilize these BEMs. It should be noted that the smoothing effect of the regridding schemes may well effect waves longer than the grid scale and that conclusive work on this topic still remains to be done. For the finite-order time domain BEMs the grid does not change in time, and thus regridding techniques cannot be used to eliminate instabilities. Furthermore, at least for the finite order models, the wiggles tend to grow exponentially in time, eventually dominating over any "physical" waves in the model. Therefore, methods such as "filtering" or "smoothing" are often used to deal with short wave instabilities in the finite order models. However, smoothing and filtering are no more than treating the symptoms of an unstable model, and thus should be applied with caution. Obviously, it would be better to solve the problem by constructing a stable model where neither smoothing nor filtering is needed to obtain stable results. In order to investigate the problem of instability a method for stability analysis is discussed in the following.

Stability analysis

In a previous workshop Vada and Nakos (1993) presented a stability analysis for a perturbation based time-domain BEM with B-spline basis functions for linear ship motions in waves. The analysis assumes third order B-splines (piecewise quadratic polynomials), deep water, and that the underlying current direction is parallel to the grid panels. Also, as in most other stability analyses, it is assumed that the free surface extends infinitely in the two horizontal directions and that the free surface grid is uniform and rectangular. Thus, the effects of ships and truncation boundaries on the free surface are ignored in the stability analysis. Recently the analysis by Vada and Nakos has been extended by Büchmann (1999) to include higher order B-spline basis functions as well as effects of finite water depth and the current direction to intersect panels obliquely. These analyses are quite lengthy, and a central part is the discrete spatial Fourier transform of the basis functions, the Boundary Integral Equation and the two linearized boundary conditions on the free surface. Also the discrete temporal Fourier transform (the so-called z-transform) of the time integration scheme is needed. After some rather lengthy manipulations a stability criterion is typically obtained by numerically solving a large number of eigenvalue problems. The main conclusions of both analyses are that for a specific model with a given current and mesh size, there exists a critical time step size, where the model will be stable in time for any choice of time step size smaller than this. The obtained stability conditions are basically of the same form as the Courant condition found by Dommermuth and Yue (1987) – only the critical value of the time step size varies between the models.

[1]Department of Hydrodynamics and Water Resources, Technical University of Denmark , DK-2800 Lyngby, Denmark (buchmann@isva.dtu.dk).

[2]Danish Hydraulic Institute, Agern Allé 5, DK-2970 Hørsholm, Denmark (jes@dhi.dk).

It should be noted, however, that certain time integration schemes may lead to models that are unconditionally unstable. A classical example of this is the fully explicit Euler scheme, which also in this context yields unstable solutions for all choices of the time step size.

APPLICATION TO A TIME-DOMAIN BEM

The numerical solution to hydrodynamic problems with a free surface often requires regridding, smoothing or filtering to be applied both in fully non-linear models (see *e.g.* Boo and Kim, 1997; Ferrant, 1997; Celebi *et al.*, 1998) and in perturbation based models (see *e.g.* Vada and Nakos, 1993; Kim *et al.*, 1997; Büchmann *et al.*, 1998). Since the stability analysis outlined in the previous section assumes that the model is perturbation based (first order actually), a comparison will be made to a new perturbation based time-domain BEM with B-spline basis functions. As excellently pointed out by Dommermuth and Yue (1987) the control over the minimum mesh size is at least a necessary condition for stability. However, even for fully non-linear model where the nodes follow prescribed paths (thus effectively controlling the mesh size) a filtering procedure may well prove necessary to eliminate wiggles (see *e.g.* Ferrant, 1997), *i.e.* control over minimum mesh size is not a sufficient condition for stability.

A linearized time-domain BEM with B-spline basis functions is considered. The elevation η, the potential ϕ and the normal derivative of the potential ϕ_n are approximated on the boundary by B-spline basis functions b_j. Thus at any position x on the boundary, η and ϕ may be written as linear combinations of the basis functions as *e.g.*

$$\eta(x,t) = \sum_{j=1}^{N} \tilde{\eta}_j(t)\, b_j(x) \quad , \quad \phi(x,t) = \sum_{j=1}^{N} \tilde{\phi}_j(t)\, b_j(x) \tag{1}$$

Here t denotes time, and $\tilde{\eta}_j$ and $\tilde{\phi}_j$ are weights of the basis functions to yield η and ϕ respectively. These approximations are used both in the boundary integral equations and in the linearized free-surface conditions (see *e.g.* Büchmann, 1999). The model is using the collocation approach defining one collocation point at the centroid of each basis function and satisfying the boundary conditions and the boundary integral equation at these points. At each time step the linearized free surface conditions are time integrated to obtain η and ϕ on the free surface at the new time level, the Neumann conditions are used to obtain ϕ_n on the remaining boundaries and finally the boundary integral equations are solved to yield the remaining unknowns on the new time level. On the free surface boundary the mixed implicit–explicit Euler scheme (see *e.g.* Vada and Nakos, 1993) is used for the time integration. Thus, the kinematic condition is updated using the explicit Euler scheme, while the dynamic condition is updated using the implicit Euler scheme. Stability analysis show that the resulting scheme is conditionally neutrally stable in time and has no spurious roots. Furthermore, as shown by Büchmann (1999), the scheme is second-order accurate in time even though both the pure Euler schemes by themselves are only first-order accurate. No filtering or smoothing are used on the free surface or anywhere else.

In the remaining part of this abstract one particular simple case will be examined in detail. Consider a fluid domain bounded by a free surface, a horizontal sea bed and four vertical truncation boundaries together forming a cube. Each of the six boundaries is discretized into 64 square panels of equal size, thus the water depth is h=$8\Delta x$, where Δx is the mesh size. Choosing third order B-spline basis functions (piecewise quadratic polynomials) results in 100 collocation points for each of the six boundaries. A current defined by $F_h \equiv U/\sqrt{g\Delta x}=0.10$ is applied in one of the main directions of the free surface panels. Here F_h is called the grid Froude number and g is the gravitational acceleration. Stability analysis show that this setup will result in a stable solution if the time step size, Δt, is chosen such that $\beta \equiv \sqrt{\Delta x}/(\Delta t \sqrt{g}) > 1.05$. For the first order velocity potential, homogeneous Neumann conditions are applied on the sea bed and at the truncation boundaries. To initiate the calculations a standing wave is starting from rest, with wave length L=$16\Delta x$ (corresponding to a non-dimensional wave number kh=π) and wave propagation directions perpendicular to the current direction.

Time series from the standing wave simulations for three different time step sizes can be seen in Figure 1. It is seen that the model is unstable for all three time step sizes, even though the stability analysis predicted the model to be stable. A longer simulation will reveal that the instabilities grow exponentially in time. Furthermore, it is noticed from the Figure that the three simulations all show exactly the same instability behavior. In fact, the model will converge temporally to a solution which is unstable in time. The reason for this behavior is the following: In order to make a stability analysis a set of assumptions were made (see above). Thus, when interpreting the result from the stability analysis it is important to note that the stability criterion obtained ($\beta > 1.05$) is a necessary (but obviously not sufficient) condition for stability. Since the oscillation period of the instability is independent of the time step size, temporal filtering seems impractical to stabilize the BEM. However, the instability occurs on a very high wave number, which makes spatial filtering feasible. These conclusions are supported by the fact that spatial filtering is utilized in many time-domain BEMs, while temporal filtering rarely is used.

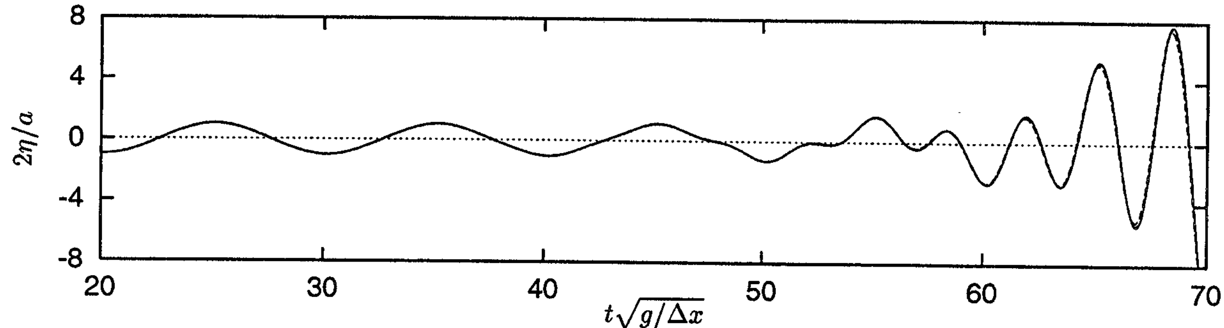

Figure 1: Time series of elevation at a collocation point near a corner of the domain using three different time step sizes corresponding to $\beta = 10$ (---), $\beta = 20$ (····) and $\beta = 50$ (—) showing a pronounced temporal instability. The results are for the chosen model problem with picewise quadratic basis functions, F_h=0.10 and initialized by a standing wave. The elevation is normalized by the initial amplitude a.

Examining the spatially discretized system of equations of the model in details, it is possible to eliminate the Neumann conditions and the boundary integral equations, rewriting the system of equations as

$$\dot{\Psi} = M\Psi + F(t) \quad , \quad \Psi = \begin{pmatrix} \tilde{\eta}^{fs} \\ \tilde{\phi}^{fs} \end{pmatrix} \tag{2}$$

The overdot denotes a time derivative, and $\tilde{\eta}^{fs}$ and $\tilde{\phi}^{fs}$ are the weights of the basis functions to yield the elevation η and the potential ϕ on the free surface. It is important to note that the real matrix M is independent of both Ψ and time. Thus, (2) represents a set of linear first-order ordinary differential equations with constant coefficients.

Introducing the temporal discretization (in this case the mixed implicit–explicit Euler scheme) the time integration procedure can be written as a difference equation on the form

$$\Psi^{(n+1)} = \widehat{M}\Psi^{(n)} + \widehat{F}^{(n)} \tag{3}$$

Eigenvalue pair number	Eigenvalue modulus	argument
1	1.0186665	±0.1892239
2	1.0186002	±0.1899729
3	1.0185307	±0.1917973
4	1.0184384	±0.1947614
(⋆)	1.0185984	±0.1907245

Table 1: The four conjugated pairs of eigenvalues with largest modulus found by numerically solving the eigenvalue problem $\widehat{M}v = v\lambda$ obtained from (3). The eigenvalue estimated from time series is also given (⋆). The results are for the chosen model problem with a time step size corresponding to β=10.

The superscripts on the basis function weight vectors denote the time step numbers. For each (complex) eigenvalue $\hat{\lambda}$ of $\widehat{M}$, (3) has an eigen-solution (solution to the homogeneous equation) where $\Psi^{(n+1)} = \hat{\lambda}\Psi^{(n)} = \hat{\lambda}^n\Psi^{(1)}$. The modulus of the eigenvalue corresponds to the magnification of the solution vector per time step taken, while the argument can be related to the period of the instability. Thus, if $\widehat{M}$ has an eigenvalue with modulus larger than unity, then the corresponding eigen-solution will grow exponentially in time, exactly as observed in Figure 1. This means that the eigenvalue with largest modulus can be estimated from a long time series, much like using the "power method" to obtain an eigenvalue. However, in this particular case the eigenvalues with maximum modulus are very closely spaced (see Table 1). Thus, the eigenvalue obtained from the time series may depend on the amount of energy initially being on each eigen-solution. In the present case the time series follow closely the eigen-solution corresponding to the eigenvalue with second largest modulus. By changing the initial conditions to a current direction parallel to the standing wave profile would bring out the eigen-solution corresponding to the eigenvalue with largest modulus. In principle the "largest eigen-solution" will dominate if time gets large enough (independent of the initial conditions), but that may not happen before an "overflow error" is encountered. A comparison of the obtained surface elevation from the time-domain model (for a large value of t) with the eigenvector (eigen-solution) corresponding to the eigenvalue with second largest modulus show very good agreement (results not shown here).

In order to eliminate all effects of the time integration scheme, (2) is once more considered. For each (complex) eigenvalue λ of M, (2) has an eigen-solution $\Psi(t) = \Psi(0)e^{\lambda t}$. Thus, if M has an eigenvalue with positive real part, then that solution will grow exponentially in time. For the present model case the eigenvalues of M have been found numerically (see Figure 2). The eigenvalues with negative real part corresponding to temporally damped (strongly stable) eigen-solutions are not shown. Shown are only the eigenvalues with zero or positive real part, corresponding to eigen-solutions which are respectively periodic or unstable in time. The eigenvalues with negative imaginary part can be obtained as complex conjugates of the shown eigenvalues.

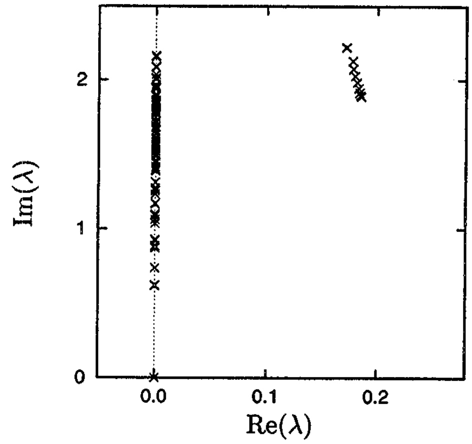

Figure 2: Eigenvalues of M in (2) for the chosen model problem.

Examining more closely the eigenvalue with second largest real part ($\lambda = 0.1842960 + i1.8969$) the expected unstable solution can be obtained once again. In particular the growth rate $\exp(\text{Re } \lambda) = 1.20237$ (over one non-dimensional time unit), is very close to $(\hat{\lambda}_{(*)})^{\beta} = 1.0185984^{10} = 1.20235$, which was observed in the time series (see also Table 1). Also the angular frequency of the instability $\text{Im}(\lambda) = 1.8969$ agrees fairly well with the time series estimate $\beta\arg(\hat{\lambda}_{(*)}) = 10 \cdot 0.19072 = 1.9072$. Thus, it is evident that the instabilities observed in the time series (see Figure 1) is not a feature of the chosen time integration scheme, but is actually a property of the spatially discretized equations. Thus, choosing other time integration schemes, such as higher-order Adams or Runge-Kutta schemes, the same instabilities may well be observed.

Some time integration schemes (such as *e.g.* the purely implicit Euler scheme) have the property of stabilizing eigen-solutions corresponding to eigenvalues in certain parts of the positive real half plane. However, these schemes are often implicit in type, and thus the solution of the boundary integral equations are required several times at each time step. Thus, these methods may well be too demanding with respect to computing time. Also, since the stability analysis did not predict this kind of instability, a new stability analysis which includes the non-uniformity of the free surface as well as Neumann boundary effects needs to be made, in order to find a stability criterion for these methods in connection with the time-domain BEMs. Obviously, such an analysis will be very complicated in general and is well beyond the scope of this work. Finding the eigenvalues of M in (2) to obtain a stability criterion is possible, but this alternative is clearly very expensive in CPU time.

Vada and Nakos (1993) as well as Kim *et al.* (1997) suggest that the instabilities observed in their models (and not predicted by their stability analysis) are caused by a wave number with zero group velocity (a "resonant mode"), resulting in a wave where the energy cannot be radiated away. Thus, the instabilities should be caused by the external forcing of the problem. However, this argument fails to explain the observed exponential growth of the instabilities in time. Based on the observations made in this abstract, it is conjectured that the instabilities are in fact due to effects in the spatially discretized models – these effects not being included in the respective stability analyses.

REFERENCES

Boo, S. Y. and Kim, C. H., (1997). Nonlinear irregular waves and forces on truncated vertical cylinder in a numerical wave tank. *Proceedings of the 7th International Offshore and Polar Engineering Conference*, vol. 3, 76–84, Honolulu, Hawaii, USA.

Bunnik, T. H. J. and Hermans, A. J., (1998). Stability analysis for solving the 3D unsteady free-surface condition with raised panels. *Proceedings of the 13th International Workshop on Water Waves and Floating Bodies*, Alphen aan den Rijn, The Netherlands.

Büchmann, B., Skourup, J. and Cheung, K. F., (1998). Run-up on a structure due to second-order waves and a current in a numerical wave tank. *Applied Ocean Research*, **20**(5), 297–308.

Büchmann, B., (1999). Accuracy and stability of time-domain boundary element models based on B-splines. *In preparation.*

Celebi, M. S., Kim, M. H. and Beck, R. F., (1998). Fully nonlinear 3-D numerical wave tank simulation. *Journal of Ship Research*, **42**(1), 33–45.

Dommermuth, D. G. and Yue, D. K. P., (1987). Numerical simulation of nonlinear axisymmetrical flows with a free surface. *Journal of Fluid Mechanics*, **178**, 195–219.

Ferrant, P., (1997). Simulation of strongly nonlinear wave generation and wave-body interactions using a 3-D MEL model. *Proceedings of the 21st ONR Symposium on Naval Hydrodynamics*, 93–109, Trondheim, Norway, 1996.

Kim, Y., Kring, D. C. and Sclavounos, P. D., (1997). Linear and nonlinear interactions of surface waves with bodies by a three-dimensional Rankine panel method. *Applied Ocean Research*, **19**(5-6), 235–249.

Longuet-Higgins, M. S. and Cokelet, E. D., (1976). The deformation of steep surface waves on water, I. A numerical method of computation. *Proceedings of the Royal Society of London, Series A*, **350**, 1–26.

Vada, T. and Nakos, D. E., (1993). Time-marching schemes for ship motion simulations. *Proceedings of the 8th International Workshop on Water Waves and Floating Bodies*, 155–158, St. John's, Newfoundland, Canada.