Numerical Investigation of Shallow-Water Wave Equations of BOUSSINESQ Typ

TAO JIANG & RUPERT HENN
Institute of Ship Technology
Mercator University, Duisburg
Germany

Introduction

In recent papers by JIANG (1998) and JIANG and SHARMA (1998) it has been shown that shallow-water wave equations of BOUSSINESQ type can be successfully used to simulate ship waves in shallow water. For a slender ship the well-established technique of matched asymptotic expansions can be applied to approximate the near-ship flow. For a flat ship a pressure distribution proportional to the local draft can be used to approximate the ship influence on the ambient water. The resulting computer programs were recently applied to predict the wave generation of an inland passenger-ferry moving in a shallow-water channel. The body plan is reproduced in Fig. 1. The under-water ship form is charaterized by a large beam-draft-ratio ($\frac{B}{T} = 7.37$) and a small value of the slenderness parameter defined by $\delta = \frac{\sqrt{S_o}}{L} = 0.072$ with the main sectional area S_o and the length L on the water line. Fig. 2 compares the calculated wave records from the computer programs based on the above two appoximation methods with that measured. For the design speed of $F_{nh} = 0.873$ the main observations are: (i) The agreement is good near the ship (responsible for the good agreement of the wave resistance) and ahead of the ship (good prediction of the solitary waves). (ii) The remarkable agreement for the wave records near the channel wall indicates a realistic dispersion relation of the BOUSSINESQ equations and the correct implementation of the boundary condition on the wall. But the relatively large discrepancy behind the ship near the tank center may be caused by the deeply submerged transom stern which has not been explicitly treated in the present approximations.

Moreover, it was shown that calculations near the bifurcation points, defined by the transition from steady solution to the unsteady one in the subcritical speed range or by the transition vice versa in the supercritical speed range, depended on the time step-size and were sensitive to numerical filtering. Therefore, the present study focuses on the improvement of the numerical methods implemented in our computer codes. To efficiently investigate these numerical problems the one-dimensional shallow-water equations of BOUSSINESQ type are used here without loss of generality. The resulting computer program will be applied to simulate some one-dimensional wave problems, such as waves generated by a wave maker in a shallow-water channel or a pressure distribution moving over waves.

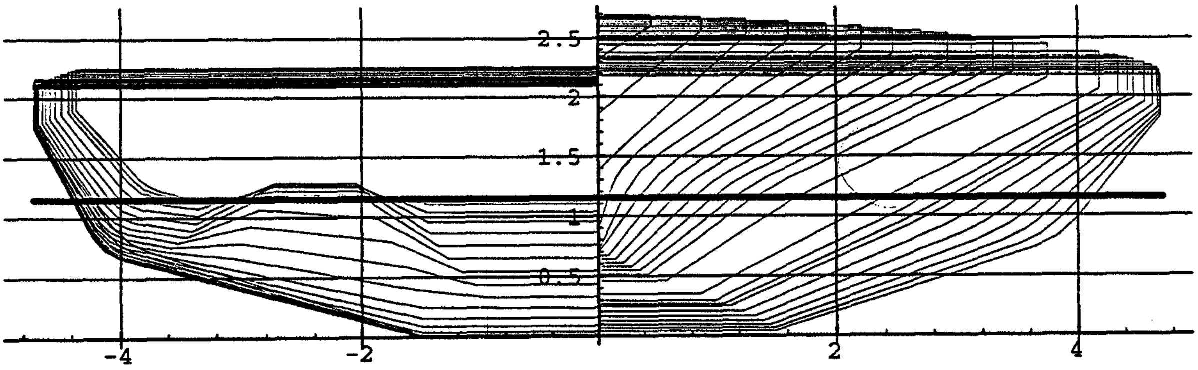

Fig. 1 Body plan of an inland passenger-ferry

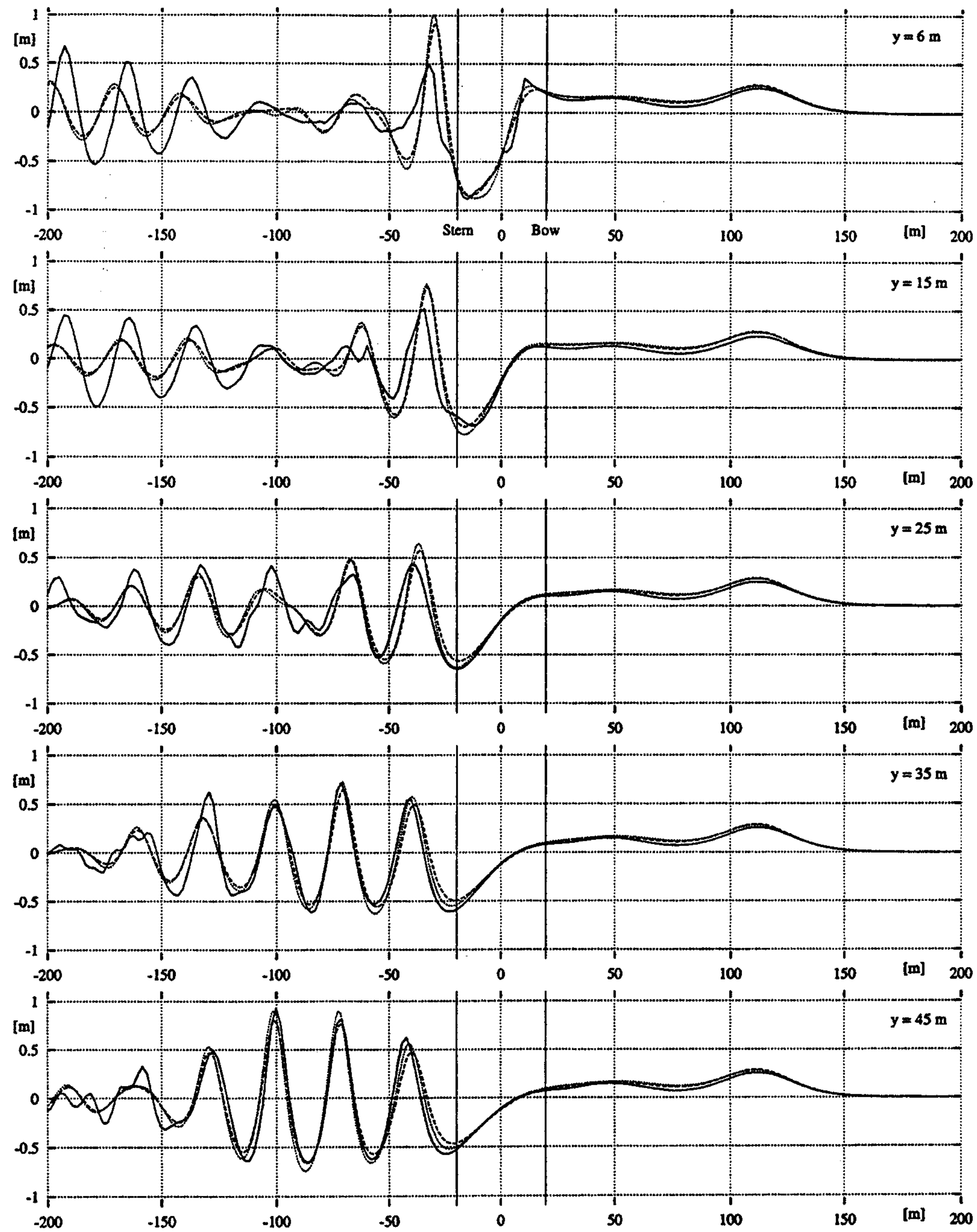

Fig. 2 Comparison of wave profiles from calculations and model measurements for an inland passenger-ferry at speed $F_{nh} = 0.873$ in shallow water channel of depth 5 m (—— measured; – – – calculated by using slender-body theory for the near field; ······ calculated by using pressure appximation)

Modified BOUSSINESQ Equations

The basic assumptions of BOUSSINESQ equations are that the waves are weakly nonlinear and long in comparison to water depth. The former implies that wave amplitude ζ_A is small compared with water depth h_o. The latter means that water depth is small compared with wave length λ. By matching $\frac{\zeta_A}{h_o} \approx (\frac{2\pi h_o}{\lambda})^2$ the flow is governed by the classical BOUSSINESQ equations

$$h_t - Vh_x + hu_x + uh_x = 0, \tag{1}$$

$$u_t - Vu_x + uu_x + gh_x - \frac{h_o^2}{3}(u_{xxt} - Vu_{xxx}) = 0, \tag{2}$$

in a coordinate system Oxz moving with a constant speed V. The x-axis is on the quiet free surface and z-axis is positive upward. The horizontal velocity component averaged over the local water depth $h = h_o + \zeta(x,t)$ is denoted by u and the wave elevation by ζ.

To improve the dispersion relation for short waves the above BOUSSINESQ equations can be modified by adding higher order linear terms in equation (2). The modified BOUSSINESQ equations read now:

$$h_t - Vh_x + hu_x + uh_x = 0, \tag{3}$$

$$u_t - Vu_x + uu_x + gh_x - \frac{h_o^2}{3}(1 - 3C_{BQ})(u_{xxt} - Vu_{xxx}) + gh_o^2 C_{BQ} h_{xxx} = 0, \tag{4}$$

with the coefficient C_{BQ} needed to be specified. Its suitable value can be found by comparing the dispersion relation of the modified BOUSSINESQ equations

$$\frac{V^2}{gh_o} = \frac{1 - C_{BQ}(kh_o)^2}{1 + (\frac{1}{3} - C_{BQ})(kh_o)^2} = 1 - \frac{1}{3}(kh_o)^2 + \frac{1}{3}(\frac{1}{3} - C_{BQ})(kh_o)^4 - \cdots, \tag{5}$$

with that from the linear wave theory

$$\frac{V^2}{gh_o} = \frac{\tanh kh_o}{kh_o} = 1 - \frac{1}{3}(kh_o)^2 + \frac{2}{15}(kh_o)^4 - \cdots, \tag{6}$$

where $k = \frac{2\pi}{\lambda}$ is the wave number. An approximation of order $O[(kh_o)^4]$ leads to the suitable value $C_{BQ} = -\frac{1}{15}$. An optimum value of $C_{BQ} = -0.057165$ can be found in the approximation range of kh_o from 0 to π. To demonstrate the influence of C_{BQ} on the dispersion relation, Fig. 3 shows the dispersion relations of the linearized classical BOUSSINESQ equations with three different values of C_{BQ} and compares them with that from the linear wave theory for finite-depth water. It can be seen that the modified Boussinesq equations are valid for a wave length down to twice the water depth without noticeable errors.

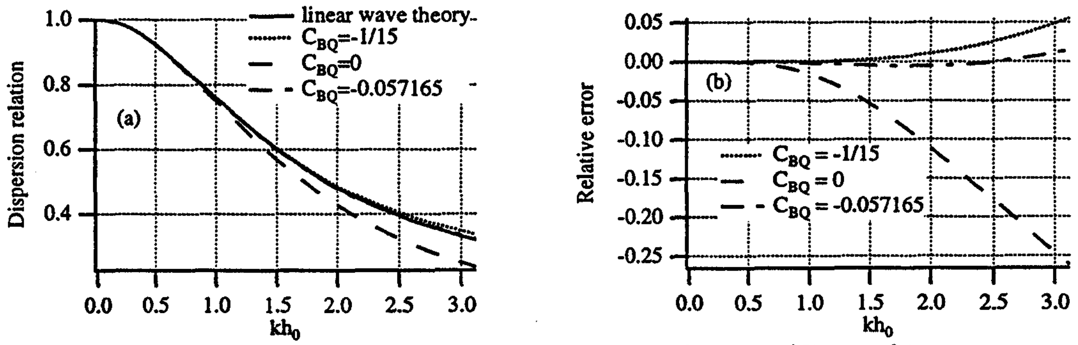

Fig. 3 Comparison of the dispersion relations of linear ship waves

Numerical Solution

The Boussinesq equations (3) and (4) are now discretized in time and space by using the Crank-Nicholson scheme. To show this procedure the governing equations are represented by the following vector equation

$$\mathbf{Q}_t + \mathbf{M}_x \mathbf{Q}_x + \mathbf{M}_{txx} \mathbf{Q}_{txx} + \mathbf{M}_{xxx} \mathbf{Q}_{xxx} = 0, \tag{7}$$

with the state vector and matrices

$$\mathbf{Q} = \begin{pmatrix} h \\ u \end{pmatrix}, \qquad \mathbf{M}_x = \begin{bmatrix} -V+u & h \\ g & -V+u \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} -V & h \\ g & -V \end{bmatrix}$$

$$\mathbf{M}_{txx} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{h_o^2}{3}(1 - 3C_{BQ}) \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{xxx} = \begin{bmatrix} 0 & 0 \\ C_{BQ} g h_o^2 & \frac{h_o^2}{3}(1 - 3C_{BQ})V \end{bmatrix}.$$

Now by defining the difference operators

$$\bar{\delta}\mathbf{Q} = \frac{1}{2}(\mathbf{Q}^{n+1} + \mathbf{Q}^n), \qquad \delta_t\mathbf{Q} = \frac{1}{\Delta t}(\mathbf{Q}^{n+1} - \mathbf{Q}^n),$$

$$\delta_x\mathbf{Q} = \frac{1}{2\Delta x}(\mathbf{Q}_{i+1} - \mathbf{Q}_{i-1}), \qquad \delta_{xx}\mathbf{Q} = \frac{1}{\Delta x\Delta x}(\mathbf{Q}_{i+1} - 2\mathbf{Q}_i + \mathbf{Q}_{i-1}),$$

the continuous BOUSSINESQ equations turn into a finite difference form

$$\begin{aligned}
&\mathbf{I}\mathbf{Q}_i^{n+1} \\
&+\tfrac{\Delta t}{4\Delta x}\mathbf{M}_x^{n+\frac{1}{2}}(\mathbf{Q}_{i+1}^{n+1} - \mathbf{Q}_{i-1}^{n+1}) \\
&+\tfrac{1}{\Delta x^2}(\mathbf{M}_{txx} - \tfrac{\Delta t^2}{24}\mathbf{A}\mathbf{A}) \\
&(\mathbf{Q}_{i+1}^{n+1} - 2\mathbf{Q}_i^{n+1} + \mathbf{Q}_{i-1}^{n+1}) \\
&+\tfrac{\Delta t}{4\Delta x^3}(\mathbf{M}_{xxx} - \tfrac{\Delta t^2}{8}\mathbf{M}_x^{n+\frac{1}{2}}\mathbf{A}\mathbf{A} - \tfrac{\Delta x^2}{6}\mathbf{M}_x^{n+\frac{1}{2}}) \\
&(\mathbf{Q}_{i+2}^{n+1} - 2\mathbf{Q}_{i+1}^{n+1} + 2\mathbf{Q}_{i-1}^{n+1} - \mathbf{Q}_{i-2}^{n+1})
\end{aligned}
=
\begin{aligned}
&\mathbf{I}\mathbf{Q}_i^{n} \\
&-\tfrac{\Delta t}{4\Delta x}\mathbf{M}_x^{n+\frac{1}{2}}(\mathbf{Q}_{i+1}^{n} - \mathbf{Q}_{i-1}^{n}) \\
&+\tfrac{1}{\Delta x^2}(\mathbf{M}_{txx} - \tfrac{\Delta t^2}{24}\mathbf{A}\mathbf{A}) \\
&(\mathbf{Q}_{i+1}^{n} - 2\mathbf{Q}_i^{n} + \mathbf{Q}_{i-1}^{n}) \\
&-\tfrac{\Delta t}{4\Delta x^3}(\mathbf{M}_{xxx} - \tfrac{\Delta t^2}{8}\mathbf{M}_x^{n+\frac{1}{2}}\mathbf{A}\mathbf{A} - \tfrac{\Delta x^2}{6}\mathbf{M}_x^{n+\frac{1}{2}}) \\
&(\mathbf{Q}_{i+2}^{n} - 2\mathbf{Q}_{i+1}^{n} + 2\mathbf{Q}_{i-1}^{n} - \mathbf{Q}_{i-2}^{n}),
\end{aligned}
\tag{8}$$

with the matrix

$$\mathbf{M}_x^{n+\frac{1}{2}} = \mathbf{M}_x(\mathbf{Q}_i^{n+\frac{1}{2}}).$$

By explicitly evaluating the state vector $\mathbf{Q}^{n+\frac{1}{2}}$ using its value at time-step n

$$\begin{aligned}
\mathbf{Q}^{n+\frac{1}{2}} &= \mathbf{Q}^n - \frac{1}{1!}(\frac{\Delta t}{2})[\mathbf{M}_x\delta_x\mathbf{Q}^n + (\mathbf{M}_{xxx} - \mathbf{M}_{txx}\mathbf{A})\delta_x\delta_{xx}\mathbf{Q}^n] \\
&+\frac{1}{2!}(\frac{\Delta t}{2})^2\mathbf{A}\mathbf{A}\delta_{xx}\mathbf{Q}^n - \frac{1}{3!}(\frac{\Delta t}{2})^3\mathbf{A}\mathbf{A}\mathbf{A}\delta_x\delta_{xx}\mathbf{Q}^n,
\end{aligned} \tag{9}$$

the finite-difference form in (8) represents a linear algebraic equation system.

Unified Conditions on the Truncating Boundary

For the given initial conditions ($h = h_o$ and $u = 0$) the solution of the above linear algebraic equation system is defined by the boundary conditions. The general conditions on the truncating boundaries can be unified as follows:

$$(1-r)s\sqrt{\frac{g}{h_{\text{in}}}}h + u = (1-r)s\sqrt{\frac{g}{h_{\text{in}}}}h_{\text{in}} + u_{\text{in}}, \tag{10}$$

$$h_t - s\sqrt{\frac{h}{g}}u_t + (-V + u - s\sqrt{gh})(h_x - s\sqrt{\frac{h}{g}}u_x) = 0, \tag{11}$$

with $s = 1$ for down-stream truncation , $s = -1$ for up-stream truncation and $s = 0$ for a wave maker; $r = 0$ for a no-reflection boundary, $r = 1$ for a total-reflection boundary. $h_{\text{in}}, u_{\text{in}}$ stand for incident values. The time and space discretizations of equations (10) and (11) follow again from the Crank-Nicholson scheme but now using only the one-sided space-difference towards the calculation domain. To suppress numerical oscillations the local and global filtering techniques are implemented, see e.g. SCHRÖTER (1995).

References

Jiang, T., "Investigation of Waves Generated by Ships in Shallow Water," Proceedings of the 22th Symposium on Naval Hydrodynamics, Washington, DC, 1998.

Jiang, T., and Sharma, S.D., "Wavemaking of Flat Ships at Transcritical Speeds," Proceedings of the 19th Colloquium on Ship and Ocean Technology, Duisburg,1998.

Schröter, A., "Nonlinear Time-Discretized Seaway Simulation in Shallow and Deep Water (in German)," Institut für Strömungsmechanik and Elektronisches Rechnen im Bauwesen der Universität Hannover, Report No. 42, 1995.