# A Hierarchical Interaction Theory for Wave Forces on a Great Number of Buoyancy Bodies


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## 1. Introduction

Very large floating structures (VLFSs) are considered for a floating airport in sheltered areas. Although many studies have been recently made on the pontoon type VLFS, there can be a column-supported type VLFS, comprised of a thin upper deck and a great number of buoyancy bodies.

In this column-supported type, besides the upper deck is flexible due to relatively small rigidity, hydrodynamic interactions among a great number of columns are important in evaluating the diffraction and radiation forces. It is said that the number of columns will be as large as more than 10,000 and conventional calculation methods cannot be used owing to a huge amount of computer memory and computation time.

In this paper, a new hierarchical interaction theory is presented, which is an extension of Kagemoto & Yue's (1986) interaction theory. Theoretically, no matter how many columns are used, the present theory can be applied with relatively small computation time and hydrodynamic interactions can be taken into account rigorously in the framework of linearized potential theory.

In connection with hydrodynamic interactions, some researchers have recently studied the so-called trapped-wave phenomena among a certain number of cylinders; e.g. Maniar & Newman (1997). These trapped-wave phenomena occur at some specific frequencies when the wavelength is of the same order as the distance between the centerlines of adjacent cylinders. The present paper provides computations for those phenomena, including the wave pattern around column-supported structures with 1280 and 5120 equally-spaced circular cylinders.

## 2. Formulation

We consider a column-supported type VLFS, including a great number of buoyancy columns which are identical and equally spaced. The geometry of each column can be arbitrary but the column considered here is a truncated circular cylinder with radius $a$ and draft $d$. The centerlines of adjacent cylinders are separated by a distance $2s$ in both $x$- and $y$-axes of a Cartesian coordinate system, where $z=0$ is the plane of the undisturbed free surface and the water depth is constant at $z=h$. Incident plane waves propagate in the direction with angle $\beta$ relative to the $x$-axis. In addition to the global coordinate system, we will use a local cylindrical coordinate system $(r_j, \theta_j, z)$, with the origin placed at the center of the $j$-th cylinder, $(x_j, y_j, 0)$.

Time-harmonic motions of small amplitude are considered, with the complex time dependence $e^{i\omega t}$ applied to all first-order oscillatory quantities. The boundary conditions on the body and free surface are linearized, and the potential flow is assumed.

We then express the velocity potential in the form

$$\Phi = \frac{gA}{i\omega}\left\{ \Phi_I(x,y,z) + \Phi_S(x,y,z) \right\} + \sum_{k=1}^{\infty} i\omega X_k\, \Phi_k(x,y,z) \tag{1}$$

where $A$ is the amplitude of an incident wave, $\omega$ is the circular frequency, and $g$ is the gravitational acceleration.

$\Phi_I$ and $\Phi_S$ in (1) represents the incident-wave potential and the scattered potential, respectively, and the sum, $\Phi_D \equiv \Phi_I + \Phi_S$, is referred to as the total diffraction potential.

In the radiation component, suffix $k$ refers to the $k$-th mode of motion, which includes not only rigid-body motion but also a set of "generalized" modes to represent elastic deflections of an upper deck. $X_k$ denotes the complex amplitude of each mode, which may be determined by solving the vibration equation of the upper deck.

In what follows, attention is focused only on the diffraction problem for clarity of explanation, but the radiation problem can be solved with the same concept (Kashiwagi, 1998).

## 3. Diffraction Problem

### 3.1 Diffraction characteristics of a single body

In the interaction theory among a large number of bodies, it is prerequisite to solve the diffraction problem of the $j$-th body in a set of "generalized" incident waves defined by

$$\{\psi_I^j\} = \left\{ \begin{array}{l} Z_0(z) J_p(k_0 r_j) e^{-ip\theta_j} \\ Z_n(z) I_p(k_n r_j) e^{-ip\theta_j} \end{array} \right\} \tag{2}$$

where $p = 0, \pm 1, \pm 2, \ldots, \pm\infty$; $n = 1, 2, \ldots, \infty$; and

$$Z_0(z) = \frac{\cosh k_0(z-h)}{\cosh k_0 h}\ ,\ Z_n(z) = \frac{\cos k_n(z-h)}{\cos k_n h} \tag{3}$$

$$\frac{\omega^2}{g} \equiv K = k_0 \tanh k_0 h = -k_n \tan k_n h \tag{4}$$

$J_p$ and $I_p$ in (2) denote the first kind of Bessel and modified Bessel functions, respectively.

The diffraction problem for the above incident waves can be solved for arbitrary shaped bodies, using boundary element method. Resultant scattered potential can be written in the form

$$\{\varphi_S^j\} = \left[B_j\right]^T \{\psi_S^j\}\ ;\quad \{\psi_S^j\} = \left\{ \begin{array}{c} Z_0(z) H_m^{(2)}(k_0 r_j)\, e^{-im\theta_j} \\ Z_n(z) K_m(k_n r_j)\, e^{-im\theta_j} \end{array} \right\} \tag{5}$$

with $m = 0, \pm 1, \pm 2, \ldots, \pm\infty$, and $n = 1, 2, \ldots, \infty$.

$H_m^{(2)}$ and $K_m$ are the second kind of Hankel and modified Bessel functions, respectively. The coefficient matrix $\left[B_j\right]$ is referred to as the diffraction characteristics matrix of the $j$-th body, and $\left[B_j\right]^T$ denotes its transpose.

The vector of wave forces in the $k$-th direction exerted by the generalized incident waves may be computed at the same time and expressed in the form

$$\{E_k^j\} = \iint_{S_j} \{\psi_I^j + \varphi_S^j\}\, n_k\, dS = \iint_{S_j} \{\varphi_D^j\}\, n_k\, dS \tag{6}$$

where $S_j$ denotes the mean wetted surface of the $j$-th body.

### *3.2 Hierarchical interaction theory*

The actual bodies in an array with equal spacing are referred to as bodies at level one in the present theory. A number of level-one bodies are grouped to form a fictitious body, which is at level two, and several fictitious bodies are grouped further to form a bigger fictitious body at level three. Repeating this hierarchical treatment makes it possible to view the interactions among a large number of bodies as a succession of simpler interactions among a smaller number of bodies. However, for explaining essence of the theory, it may be enough to consider only two levels, i.e. $\ell = 2$.

Writing the incident-wave potential in terms of a polar coordinate system of fictitious body $i$ at level $\ell$, we obtain the following:

$$\Phi_I = e^{-ik_0(x_i \cos\beta + y_i \sin\beta)} \sum_{p=-\infty}^{\infty} e^{ip(\beta - \frac{\pi}{2})} \left\{ Z_0(z) J_p(k_0 r_i)\, e^{-ip\theta_i} \right\} \tag{7}$$

With the vector of generalized incident waves defined by (2), $\Phi_I$ can be expressed as $\Phi_I = \{a^i\}^T \{\psi_I^i\}$, where $\{a^i\}$ is the vector of coefficients which may be explicitly given from (7).

According to Kagemoto & Yue's (1986) interaction theory, not only the incoming wave from outside but also scattered waves due to other bodies must be viewed as incident waves upon the body under consideration. Thus, utilizing the coordinate transformation matrix, $\left[T_{ij}\right]$, the total incident-wave potential on body $i$ at level $\ell$ is written as

$$\phi_{I,\ell}^i = \left( \{a^i\}^T + \sum_{\substack{n=1 \\ n \neq i}}^{N_\ell} \{A_{S,\ell}^n\}^T \left[T_{ni}^\ell\right] \right) \{\psi_{I,\ell}^i\} \tag{8}$$

where the number of fictitious bodies at level $\ell$ is assumed to be $N_\ell$, and $\{A_{S,\ell}^i\}$ is the vector of unknown coefficients of the scattered potential due to body $i$.

Assuming that the diffraction characteristics of a fictitious body $i$ at level $\ell$ are obtained and expressed with the matrix $\left[B_{i,\ell}\right]$, the following relation can be established:

$$\phi_{S,\ell}^i = \left( \{a^i\}^T + \sum_{\substack{n=1 \\ n \neq i}}^{N_\ell} \{A_{S,\ell}^n\}^T \left[T_{ni}^\ell\right] \right) \left[B_{i,\ell}\right]^T \{\psi_{S,\ell}^i\} = \{A_{S,\ell}^i\}^T \{\psi_{S,\ell}^i\} \tag{9}$$

Therefore, one can obtain a linear set of equations for the unknown coefficients, $\{A_{S,\ell}^i\}$, in the form

$$\{A_{S,\ell}^i\} - \left[B_{i,\ell}\right] \sum_{\substack{n=1 \\ n \neq i}}^{N_\ell} \left[T_{ni}^\ell\right]^T \{A_{S,\ell}^n\} = \left[B_{i,\ell}\right] \{a^i\}\ ,\quad i = 1 \sim N_\ell \tag{10}$$

However, in reality, the matrix $\left[B_{i,\ell}\right]$ is unknown, because the level $\ell$ is fictitious. To determine this matrix, the diffraction problem for the generalized incident waves, $\{\psi_{I,\ell}^i\}$, needs to be considered.

A fictitious body at level $\ell$ includes $N_{\ell-1}$ bodies at level $\ell-1$ (which are actual bodies here). Thus we must consider again the interactions among those bodies. The local (or downward) expansion of $\{\psi^i_{I,\ell}\}$ about the origin of body $j$ at level $\ell-1$ can be found by Graf's addition theorem:

$$J_m(k_0 r_i)\, e^{-im\theta_i} = \sum_{p=-\infty}^{\infty} J_{m-p}(k_0 L_{ij})\, e^{-i(m-p)\alpha_{ij}} \{ J_p(k_0 r_j)\, e^{-ip\theta_j} \} \tag{11}$$

$$I_m(k_n r_i)\, e^{-im\theta_i} = \sum_{p=-\infty}^{\infty} I_{m-p}(k_n L_{ij})\, e^{-i(m-p)\alpha_{ij}} \{ I_p(k_n r_j)\, e^{-ip\theta_j} \} \tag{12}$$

where $L_{ij}$ is the distance between $O_i$ and $O_j$ (the origins of body $i$ and body $j$, respectively) and $\alpha_{ij}$ is the azimuth angle of $O_j$ when viewed from $O_i$. These relations can be expressed in the form of $\{\psi^i_{I,\ell}\} = \left[ I^{\ell-1}_{ij} \right] \{\psi^j_{I,\ell-1}\}$.

Then, as in (8), the total incident wave potentials on body $j$ at level $\ell-1$ are written as

$$\{\varphi^j_{I,\ell-1}\} = \Big( \big[I^{\ell-1}_{ij}\big] + \sum_{\substack{n=1\\ n\neq j}}^{N_{\ell-1}} \big[A^n_{S,\ell-1}\big]^T \big[T^{\ell-1}_{nj}\big] \Big) \{\psi^j_{I,\ell-1}\} \tag{13}$$

As shown in subsection 3.1, the diffraction characteristics of a single body can be given by the matrix $[B_j]$. Therefore, in the same manner as in obtaining (10), a linear system of simultaneous equations can be derived for the coefficient matrix $[A^j_{S,\ell-1}]$, in the form

$$\big[A^j_{S,\ell-1}\big] - \big[B_{j,\ell-1}\big] \sum_{\substack{n=1\\ n\neq j}}^{N_{\ell-1}} \big[T^{\ell-1}_{nj}\big]^T \big[A^n_{S,\ell-1}\big] = \big[B_{j,\ell-1}\big] \big[I^{\ell-1}_{ij}\big]^T , \quad j = 1 \sim N_{\ell-1} \tag{14}$$

Numerical solutions of (14) completes the diffraction problem at level $\ell-1$. Then if we consider an outer-field expression of the corresponding scattered potential of $N_{\ell-1}$ bodies, the diffraction characteristics of a fictitious body may be given. For that purpose, the multipole (or upward) expansion of $\{\psi^j_{S,\ell-1}\}$ about the origin of body $i$ at level $\ell$ must be considered, which can be found by the following Graf's addition theorem:

$$H^{(2)}_m(k_0 r_j)\, e^{-im\theta_j} = \sum_{p=-\infty}^{\infty} (-1)^{m-p} J_{m-p}(k_0 L_{ji})\, e^{-i(m-p)\alpha_{ji}} \{ H^{(2)}_p(k_0 r_i)\, e^{-ip\theta_i} \} \tag{15}$$

$$K_m(k_n r_j)\, e^{-im\theta_j} = \sum_{p=-\infty}^{\infty} I_{m-p}(k_n L_{ji})\, e^{-i(m-p)\alpha_{ji}} \{ K_p(k_n r_i)\, e^{-ip\theta_i} \} \tag{16}$$

These relations can be expressed as $\{\psi^j_{S,\ell-1}\} = \left[ M^\ell_{ji} \right] \{\psi^i_{S,\ell}\}$.

Therefore, collecting the contributions from all bodies inside a fictitious body, the vector of scattered potential can be found as follows:

$$\sum_{j=1}^{N_{\ell-1}} \big[A^j_{S,\ell-1}\big]^T \{\psi^j_{S,\ell-1}\} = \sum_{j=1}^{N_{\ell-1}} \big[A^j_{S,\ell-1}\big]^T \big[M^\ell_{ji}\big] \{\psi^i_{S,\ell}\} \equiv \big[B_{i,\ell}\big]^T \{\psi^i_{S,\ell}\} \tag{17}$$

Substituting the matrix thus obtained, $[B_{i,\ell}]$, into (10) determines the coefficient vector of the scattered potential at level $\ell$; which completes the entire flow field.

### *3.3 Wave exciting force*

Since fundamental wave forces due to each component of the generalized incident waves are already computed and given by (6), the only thing for computing the wave-exciting force is to find the amplitude of waves impinging upon actual bodies. This can be done by simply combining (8) and (13), with the result

$$\phi^j_{I,\ell-1} = \{A^j_D\}^T \{\psi^j_{I,\ell-1}\} \tag{18}$$

where

$$\{A^j_D\}^T = \Big( \{a^i\}^T + \sum_{\substack{n=1\\ n\neq i}}^{N_\ell} \{A^n_{S,\ell}\}^T \big[T^\ell_{ni}\big] \Big) \Big( \big[I^{\ell-1}_{ij}\big] + \sum_{\substack{n=1\\ n\neq j}}^{N_{\ell-1}} \big[A^n_{S,\ell-1}\big]^T \big[T^{\ell-1}_{nj}\big] \Big) \tag{19}$$

With this notation, the linearized pressure on body $j$ in the diffraction problem is given by $p_D = -\rho g A \{A^j_D\}^T \{\varphi^j_D\}$. Therefore the total wave-exciting force in the $\ell$-th mode can be computed as

$$-\iint_{S_H} p_D\, n_k\, dS = \rho g A \sum_{j=1}^{N_B} \{A^j_D\}^T \iint_{S_j} \{\varphi^j_D\}\, n_k\, dS = \rho g A \sum_{j=1}^{N_B} \{A^j_D\}^T \{E^j_k\} \tag{20}$$

where $N_B$ is the total number of actual bodies.

## 4. Numerical Results

Numerical accuracy and convergence are checked for a square array of 64 half-immersed spheres, which are equally spaced with a distance of $s/a = 2.0$, in water of $h/a = 3.0$. The hierarchical interaction theory is tested with the highest level set to $\ell = 3$, in which $2 \times 2$ bodies are grouped at each level. Computed results are compared with corresponding results based on Kagemoto & Yue's (1986) theory. Very good agreement and thus validity of the theory is confirmed, though the details are not shown here due to the paucity of space.

Next, as an example of a large number of bodies, two cases are considered: (1) 1280 (16 rows times 80 columns) and (2) 5120 (32 rows times 160 columns) truncated circular cylinders. The draft ($d$) of cylinders is kept constant, whereas the radius ($a$) is changed so that the total waterplane area and displacement volume remain the same. Thus $d/a = 1.0$ for case (1) and 2.0 for case (2). The cylinders are equally spaced with $s/a = 2.0$ in water of $h/d = 5.0$.

Figure 1 shows the surge exciting force for case (1) in the frequency range where nearly trapped-wave phenomena can be expected. The dashed line denotes the results on body No.1 (which is at the upwave end along the 8th row) and the solid line denotes the results on body No.40 (which is at almost the center along the 8th row). We can see many peaks even in a narrow region of the wavenumber. According to Maniar & Newman (1997), occurrence of these peaks may be caused by a sequence of Neumann-type and Dirichlet-type trapped modes to be expected for a large number of equally-spaced cylinders.

The wave pattern at $Ks = 1.28$ around the structure of case (1) is shown in the left-hand side of Fig. 2. Likewise, the right-hand side of Fig. 2 shows the pattern around the structure of case (2) in the same wave. A plane wave is coming from the right-upper side. Interestingly, the amplitude is increasing along the longitudinal side of the structure, and there exist resonant waves whose crest line is perpendicular to that of the incident wave. These are connected with nearly trapped waves among a great number of cylinders. In case (2), large amplitude waves still exist downstream of the structure, but the wave pattern is different from that of case (1).

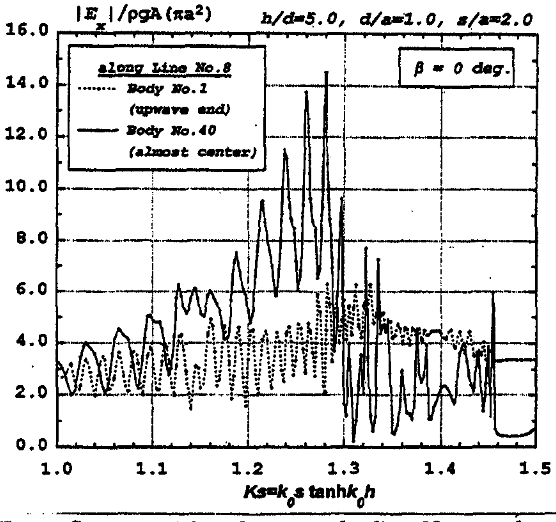


Fig. 1: Surge exciting force on bodies No.1 and No.40 along row No.8 ($N_B = 1280$)

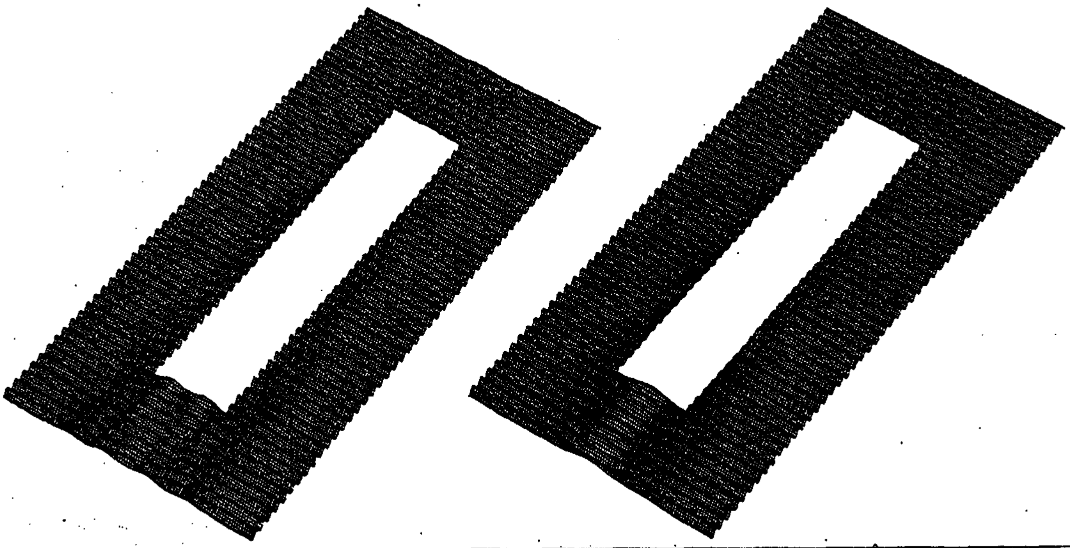

Fig. 2: Wave patterns around Model (1) ($N_B = 1280$, $Ks = 1.28$, left figure) and Model (2) ($N_B = 5120$, $Ks = 0.64$, right figure) in a wave coming from the right upper side.

## References


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