Water-wave propagation through an infinite array of cylindrical structures

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1 Introduction

A design for a floating airport in Japan involves a platform supported by an array of thousands of cylindrical legs. Kagemoto & Yue[1] proposed an approximate numerical method for the calculation of wave interaction with such an array in which the hydrodynamic characteristics of cylinders in the inner part of the array differed from one cylinder to another only by phase effects. In other words, the cylinders in the inner part are treated as if they are part of a periodic array extending to infinity in both horizontal directions. Implicit in the analysis is the assumption that waves do not decay as they propagate through an infinite array. However, this need not be the case as has been observed in the closely related problem of sound propagation through tube bundles[2]

Here, the propagation of waves through a doubly-periodic, infinite array of identical vertical cylinders extending throughout the water depth is examined. This problem is closely related to that of a particle moving in a periodic potential which has been studied extensively in solid state physics[3]. At most wave frequencies of interest propagation is indeed possible without change in amplitude although, in general, there will be a change in phase from one cylinder to another. However, for some frequency ranges, in so-called band gaps, wave propagation without amplitude change is not possible and the motion amplitude decays with distance.

2 Formulation

As the arrangement of cylinders to be considered is doubly-periodic, it is possible to confine attention to a cell in a horizontal plane that contains only a single cylinder. For a rectangular array this primitive cell is also rectangular and is illustrated in figure 1. The horizontal coordinates are illustrated and the z coordinate is directed vertically upwards with origin in the plane of the mean free surface.

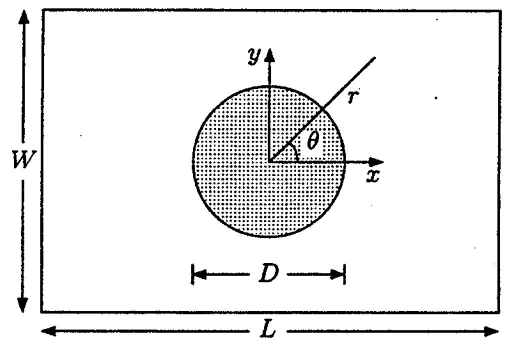

Figure 1: Definition sketch for a rectangular array.

For time-harmonic motions of angular frequency ω, the velocity potential is written as

$$\Phi(x,y,z,t) = \mathrm{Re}\{\phi(x,y)\cosh\kappa(z+h)\,\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{1}$$

where

$$(\nabla^2 + \kappa^2)\phi = 0 \tag{2}$$

throughout the fluid region, κ is the real positive root of the dispersion relation $\omega^2 = g\kappa\tanh\kappa h$ and h is the water depth. Solutions are sought in the form suggested by Bloch's theorem[3], that is

$$\phi(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\,\psi(\mathbf{r}), \tag{3}$$

where $\mathbf{r}$ is the position vector of a point in the horizontal plane and ψ has the same periodicity as the lattice. For real vectors $\mathbf{q}$ such solutions correspond to the propagation of unattenuated waves through the array, and $\mathbf{q}$ measures the change in phase as the array is traversed. If $\mathbf{q}$ has a non-zero imaginary part then there is also a decay in amplitude as a wave propagates through the array.

Let $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions respectively. For the present geometry, it may be shown that (3) is equivalent to the four independent conditions

$$\left.\begin{aligned}\phi(L/2,y) &= e^{iq_1L}\,\phi(-L/2,y),\\ \frac{\partial\phi}{\partial x}(L/2,y) &= e^{iq_1L}\,\frac{\partial\phi}{\partial x}(-L/2,y),\end{aligned}\right\}\quad -W/2 \le y \le W/2, \tag{4}$$

$$\left.\begin{aligned}\phi(x,W/2) &= e^{iq_2W}\,\phi(x,-W/2),\\ \frac{\partial\phi}{\partial y}(x,W/2) &= e^{iq_2W}\,\frac{\partial\phi}{\partial y}(x,-W/2),\end{aligned}\right\}\quad -L/2 \le x \le L/2. \tag{5}$$

The mathematical problem has been reduced to the solution of the field equation (2) within the fluid region of the cell $\{-L/2 \le x \le L/2,\ -W/2 \le y \le W/2\}$ subject to the boundary conditions (4–5) and the condition of no flow through the cylinder wall, namely

$$\frac{\partial\phi}{\partial r} = 0 \quad \text{on} \quad r = D/2, \tag{6}$$

where (r,θ) are the standard plane polar coordinates illustrated in figure 1.

Two basic approaches to the problem are used. One is to specify the wave vector $\mathbf{q}$ and then solve an eigenvalue problem for the wavenumber κ. Alternatively, κ and one component of $\mathbf{q}$ is specified and the eigenvalue problem is then to determine the second component of $\mathbf{q}$. Detailed results have been obtained using both of these approaches and, in general, the relationship between κ and $\mathbf{q}$ is very complex and much investigation remains to be done to clarify matters. However, to illustrate in a simple way some of the main features of the problem, an approximate method which applies the second of these approaches to a simplified problem is described in the following section.

3 An approximate solution

The conditions (5) are here replaced by the special case

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = \pm W/2, \tag{7}$$

which is equivalent to having solid walls at $y = \pm W/2$. This condition yields a subset of the solutions possible for $q_2 = 0$. The conditions (4) are retained in their general form. The walls can be removed if an appropriate image system is introduced in the y direction, thus the geometry of the 'channel problem' is equivalent to that in the problem originally posed. The geometry is symmetric about $y = 0$ and it will be assumed that the wavenumber is below the cut off for non-planar modes in a channel of width W so that $\kappa W < 2\pi$. The main aim is to calculate the so-called Bloch transmission coefficient

$$T_{\mathrm{B}} = e^{iq_1L} \tag{8}$$

which measures the phase change and/or the attenuation of a wave as it propagates through one cell of the array in the direction of x increasing; see equations (4).

If the cell length L satisfies $\kappa L >> 1$, then to a first approximation only plane waves propagating along the channel exist in the vicinity of $x = \pm L/2$. Thus, in the neighbourhood of $x = -L/2$

$$\phi = A_1\, e^{i\kappa x} + B_1\, e^{-i\kappa x} \tag{9}$$

and in the neighbourhood of $x = L/2$

$$\phi = A_2\, e^{i\kappa x} + B_2\, e^{-i\kappa x}, \tag{10}$$

for some complex constants A_1, A_2, B_1 and B_2. The wave with amplitude A_2 propagates away from the cylinder and is due to the transmission of A_1 and the reflection of B_2. Similarly, the wave with amplitude B_1 arises from the transmission of B_2 and the reflection of A_1. Thus

$$A_2 = TA_1 + RB_2 \quad \text{and} \quad B_1 = TB_2 + RA_1, \tag{11}$$

where R and $T = |T|e^{i\delta}$ are the reflection and transmission coefficients for a single cylinder in the channel. It may be shown that the system consisting of (11) together with the equations resulting from the application of (4), has a non-trivial solution provided

$$\cos q_1 L = \frac{\cos(\delta + \kappa L)}{|T|} \equiv f(\kappa L), \tag{12}$$

say. When $|f(\kappa L)| \leq 1$, equation (12) has only real solutions for $q_1 L$ and waves will propagate through the array with their amplitude unchanged. However, whenever $|f(\kappa L)| > 1$ solutions are of the form $q_1 L = n\pi \pm iQ$, for real $Q > 0$ and some integer n, and the Bloch transmission coefficient $T_B = (-1)^n e^{\mp Q}$. Waves that propagate in the direction of x increasing correspond to the upper sign, while the lower sign corresponds to waves propagating in the direction of x decreasing. In both case the wave attenuates as it propagates.

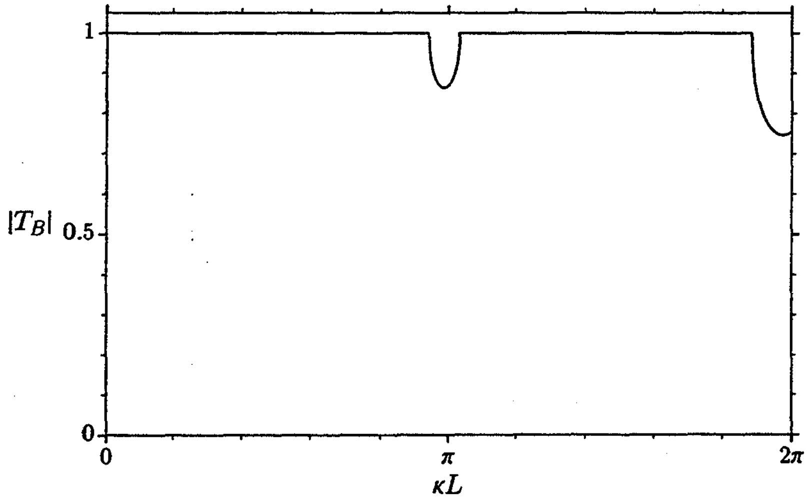

Figure 2: Approximate Bloch transmission coefficient $|T_B|$ as a function of wavenumber κL; $W/L = 1$, $D/L = 0.2$.

A typical solution for T_B is illustrated in figure 2. For simplicity, the approximations given by Martin & Dalrymple[4] are used for the reflection and transmission coefficients for a single cylinder in a channel. The parameter values have been chosen to clearly show the behaviour rather than to lie within the range of validity of the solution. For $0 \leq \kappa L < 2.96$ and $3.24 < \kappa L < 5.98$, $|T_B| = 1$ and the waves propagate through the array with amplitude unchanged; in the terminology of solid-state physics these are known as passing bands. For $2.96 < \kappa L < 3.24$ and $5.98 < \kappa L < 2\pi$, $|T_B| < 1$ so that the waves decay in amplitude as they propagate; these are known as stopping bands.

The solution for a doubly-infinite array is related to the reflection and transmission properties of an array that has finite length in one direction. Consider N infinitely long rows of cylinders situated at $x = L_m$, $m = 1, 2, \ldots, N$, with $L_{m+1} - L_m = L$ for $m = 1, 2, \ldots, N-1$. Each row may be thought of as a single cylinder with axis on $y = 0$ between channel walls at $y = \pm W/2$. A wide-spacing formalism[5] shows that R_N and T_N, the reflection and transmission coefficients for the

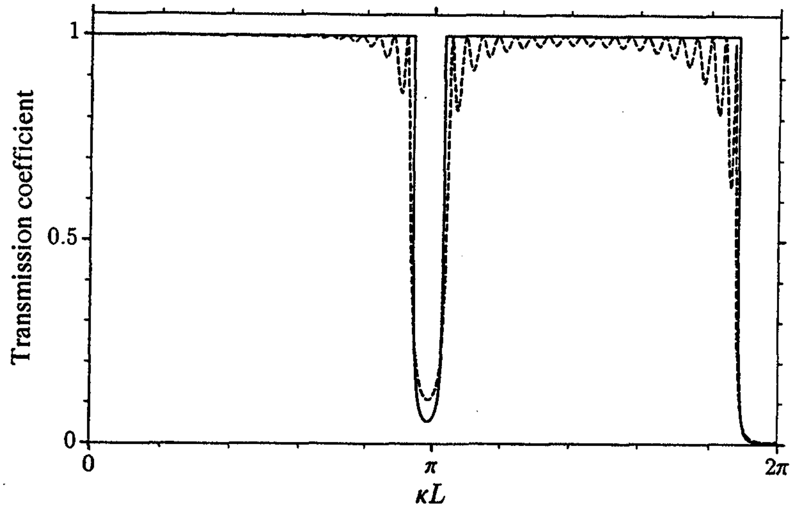

Figure 3: Transmission coefficient $|T_{20}|$ for twenty rows of cylinders (dashed line) compared with Bloch transmission coefficient $|T_B|^{20}$ (solid line) as a function of wavenumber κL; $W/L = 1$, $D/L = 0.2$.

complete array of N rows, satisfy

$$\begin{pmatrix} T_N \, e^{i(\kappa L_N + \kappa L)} \\ 0 \end{pmatrix} = S^N \begin{pmatrix} e^{i\kappa L_1} \\ R_N \, e^{-i\kappa L_1} \end{pmatrix} \quad \text{where} \quad S = \begin{pmatrix} (T - R^2/T)\, e^{i\kappa L} & R\, e^{i\kappa L}/T \\ -R\, e^{-i\kappa L}/T & e^{-i\kappa L}/T \end{pmatrix} \tag{13}$$

and this system is easily solved to determine R_N and T_N.

Results for a grating with twenty rows are displayed in figure 3. Comparison is made with $|T_B|^{20}$ which is the transmission coefficient for propagation through a distance $20L$ in the doubly infinite array. The oscillatory behaviour in $|T_{20}|$ is due to end effects for the finite number of rows. As the number of rows N in the finite array increases, $|T_N|$ approaches the value $|T_B|^N$ obtained from the doubly infinite array. Hence the Bloch transmission coefficient T_B may be used to predict the properties of a large, but finite, number of rows of cylinders.

The positions of the troughs in transmission (and hence peaks in reflection) can be explained through the phenomenon of Bragg scattering that is well-known in x-ray diffraction by a crystal. For strong overall reflection to occur the waves reflected from different rows of a grating must interfere constructively and, for the normal incidence investigated here, this occurs when $\kappa L = n\pi$, for integer n, which is clearly consistent with the results of figure 3.

References

[1] Kagemoto, H & Yue, D K P. 1986 Wave forces on a platform supported on a large number of floating legs. *Proceedings of the 5th International Conference on Offshore Mechanics and Arctic Engineering*, **1**, 206-211. ASME.

[2] Heckl, M A & Mulholland, L S. 1995 Some recent developments in the theory of acoustic transmission in tube bundles. *Journal of Sound and Vibration*, **179**, 37-62.

[3] Ashcroft, N W & Mermin, N D. 1976 *Solid State Physics.* Saunders College, Philadelphia.

[4] Martin, P A & Dalrymple R A. 1988 Scattering of long waves by cylindrical obstacles and gratings using matched asymptotic expansions. *Journal of Fluid Mechanics*, **188**, 465-490.

[5] Evans, D V. 1990 The wide-spacing approximation applied to multiple scattering and sloshing problems. *Journal of Fluid Mechanics*, **210**, 647-658.