RADIATION-DIFFRACTION PROBLEM WITH FORWARD SPEED IN A TWO-LAYER FLUID

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1. Introduction

Three-dimensional solution of ship motions with forward speed has been obtained by many investigators for a single-layer fluid of infinite depth. In this abstract, the problem is extended to a two-layer fluid of finite depth where the body is traveling in the upper layer and does not intersect the fluid interface. A two-layer fluid is often used to model density stratification due to temperature and salinity variations (small density difference) as well as a muddy bottom beneath a clean water layer (large density difference). In the latter case, the viscosity of the mud has been shown to have a negligible effect on the hydrodynamic coefficients of the body [1]. Previous solutions of the radiation-diffraction problem in a two-layer fluid are either two-dimensional [1-3] or without forward speed [4]. They show that density stratification has a significant influence on the hydrodynamic properties of the body over certain ranges of frequencies and velocities.

2. Mathematical Formulation

Let's define a rectangular coordinate system moving with the body at the mean speed U along the positive x-axis. The (x, y) plane of this system coincides with the undisturbed interface between the two fluid layers. The z-axis is positive upward. Let ρ_1, h_1, $\Phi^{(1)}$ and ρ_2, h_2, $\Phi^{(2)}$ denote the densities, depths, and total velocity potentials of the upper and lower fluid layers, respectively. Then, as in the case of a single-layer fluid, we can express $\Phi^{(m)}$ as the sum of a steady and an unsteady component as follows:

$$\Phi^{(m)}(x,y,z,t) = U\bar{\Phi}^{(m)}(x,y,z) + \Re\left\{\sum_{j=0}^{7} \zeta_j \phi_j^{(m)}(x,y,z)e^{i\sigma t}\right\}, \qquad m = 1,2. \tag{1}$$

The potentials $\Phi^{(m)}$ are determined by the following boundary-value problem:

$$\nabla^2\Phi^{(m)} = 0, \qquad 0 < z < h_1,\ m = 1, \qquad -h_2 < z < 0,\ m = 2, \tag{2}$$

$$\Phi_{tt}^{(1)} - 2U\Phi_{tx}^{(1)} + U^2\Phi_{xx}^{(1)} + g\Phi_z^{(1)} = 0, \qquad z = h_1, \tag{3}$$

$$\Phi_z^{(1)} = \Phi_z^{(2)}, \qquad z = 0, \tag{4}$$

$$\gamma(\Phi_{tt}^{(1)} - 2U\Phi_{tx}^{(1)} + U^2\Phi_{xx}^{(1)} + g\Phi_z^{(1)}) = \Phi_{tt}^{(2)} - 2U\Phi_{tx}^{(2)} + U^2\Phi_{xx}^{(2)} + g\Phi_z^{(2)}, \qquad z = 0, \tag{5}$$

$$\Phi_z^{(2)} = 0, \qquad z = -h_2, \tag{6}$$

$$\Phi_n^{(1)} = \mathbf{V}\cdot\mathbf{n}, \qquad \mathbf{x} \in S, \tag{7}$$

and an appropriate radiation condition. In the above equations, g is the gravitational acceleration, $\gamma = \rho_1/\rho_2$ is the density ratio, $\mathbf{V}$ is the local velocity of the body surface S, and $\mathbf{n}$ is the normal unit vector pointing into the body.

The potential $\bar{\Phi}^{(m)}$ is generated by the steady forward motion of the body. Its solution is addressed in [4-5]. In this paper, the body is assumed to be slender so that the steady disturbance is small and can be neglected. This simplification, though not essential, allows the unsteady potential to be obtained without solving the steady problem.

3. Incident Wave Potentials

In a two-layer fluid, time-harmonic waves can propagate at two different modes with wave-numbers k_1 and k_2, respectively. In the surface-wave mode, wavenumber k_1, the maximum displacement occurs at the free surface, and the displacements at the free surface and fluid interface are in phase. In the internal-wave mode, wavenumber k_2, the maximum displacement occurs at the interface, and the free surface and interface displacements are 180° out of phase. The dispersion relation for each mode is given by:

$$\omega_n^2 = \frac{gk_n}{2}\left[\frac{(t_1+t_2)+(-1)^{n+1}\sqrt{(t_1+t_2)^2-4\epsilon t_1 t_2(1+\gamma t_1 t_2)}}{1+\gamma t_1 t_2}\right], \qquad n=1,2, \tag{8}$$

where $t_1 = \tanh k_n h_1$, $t_2 = \tanh k_n h_2$, $\epsilon = 1-\gamma$, and ω_n is the wave frequency. If we specify the amplitude of the incident wave ζ_0 in Eqn. (1) to be the amplitude of the interface displacement, then the incident wave potentials $\phi_0^{(m)}$ can be written as:

$$\phi_0^{(1)} = \frac{i\omega_n}{k_n}\left(\frac{\omega_n^2 \tanh k_n h_1 - gk_n}{\omega_n^2 - gk_n \tanh k_n h_1}\cosh k_n z - \sinh k_n z\right) e^{i(k_n(x\cos\beta + y\sin\beta)-\sigma t)}, \qquad n=1,2, \tag{9}$$

$$\phi_0^{(2)} = -\frac{i\omega_n}{k_n}\left(\frac{\cosh(k_n(z+h_2))}{\sinh k_n h_2}\right) e^{i(k_n(x\cos\beta + y\sin\beta)-\sigma t)}, \qquad n=1,2, \tag{10}$$

where β is the incident angle measured from the positive x-axis, and σ is the encounter frequency and is related to ω_n and k_n through the following relation:

$$\sigma = \omega_n - U k_n \cos\beta. \tag{11}$$

4. Radiation and Diffraction Potentials

From Eqns. (1-7), we can obtain the following boundary-value problems for the radiation and diffraction potentials $\phi_j^{(m)}$, $j = 1,\ldots,7$.

$$\nabla^2\phi_j^{(m)} = 0, \qquad 0<z<h_1,\ m=1, \qquad -h_2<z<0,\ m=2, \tag{12}$$

$$-\sigma^2\phi_j^{(1)} - 2i\sigma U\phi_{j,x}^{(1)} + U^2\phi_{j,xx}^{(1)} + g\phi_{j,z}^{(1)} = 0, \qquad z=h_1, \tag{13}$$

$$\phi_{j,z}^{(1)} = \phi_{j,z}^{(2)}, \qquad z=0, \tag{14}$$

$$\gamma\left(-\sigma^2\phi_j^{(1)} - 2i\sigma U\phi_{j,x}^{(1)} + U^2\phi_{j,xx}^{(1)} + g\phi_{j,z}^{(1)}\right) = -\sigma^2\phi_j^{(2)} - 2i\sigma U\phi_{j,x}^{(2)} + U^2\phi_{j,xx}^{(2)} + g\phi_{j,z}^{(2)}, \qquad z=0, \tag{15}$$

$$\phi_{j,z}^{(2)} = 0, \qquad z=-h_2, \tag{16}$$

$$\phi_{7,n}^{(1)} = -\phi_{0,n}^{(1)}, \qquad \phi_{j,n}^{(1)} = i\sigma n_j + U m_j, \qquad \mathbf{x}\in\bar{S}, \tag{17}$$

where $\bar{S}$ is the mean body surface, and n_j, m_j are defined as in [6]:

$$\mathbf{n} = (n_1, n_2, n_3), \qquad \mathbf{x}\times\mathbf{n} = (n_4, n_5, n_6), \tag{18}$$

$$m_1 = m_2 = m_3 = m_4 = 0, \qquad m_5 = n_3, \qquad m_6 = -n_2. \tag{19}$$

An appropriate radiation condition is also imposed on $\phi_j^{(m)}$ for uniqueness. Once $\phi_j^{(1)}$ are solved, the hydrodynamic coefficients and exciting forces can be obtained as in the case of a single-layer fluid [6].

Figure 1: Paths of integration $L_{1,n}$ and $L_{2,n}$

5. Translating and Pulsating Green's Functions

Using Green's theorem, the unknown potentials $\phi_j^{(m)}$, $j = 1, \ldots, 7$ for a submerged body in the upper fluid layer can be expressed in terms of the translating and pulsating Green functions $G^{(m)}$ as follows:

$$\phi_j^{(m)}(x, y, z) = \int\int_{\bar{S}} \nu_j(\xi, \eta, \zeta) G^{(m)}(\xi, \eta, \zeta; x, y, z)\, d\bar{S}, \tag{20}$$

where ν_j is the source strength distribution and is determined by applying the body conditions in Eqn. (17). The resulting Fredholm integral equations of the second kind are solved using the well-known panel method.

The formulation so far is analogous to the single fluid case. The main task now is the derivation of the Green functions. These Green functions satisfy all boundary conditions of $\phi_j^{(m)}$, $j = 1, \ldots, 7$ except for the body conditions. To obtain $G^{(m)}$, we start with the unsteady potentials of a source of variable strength, starting from rest and following an arbitrary path [4]. By specifying the source strength as $\cos \sigma t$ and the source path as $\xi(t) = \xi_0 + Ut$, we can then perform the necessary integrations and limits as $t \to \infty$ to obtain the following expressions for $G^{(m)}$:

$$\begin{aligned} G^{(1)} &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{r_n} - \frac{1}{r_{1n}} \right) + \int_0^{\infty} \frac{2 \cosh kh_2 \sinh(k(z-h_1)) \sinh(k(\zeta - h_1)) J_0(kR)}{\sinh kh_1 (\cosh kh_1 \cosh kh_2 + \gamma \sinh kh_1 \sinh kh_2)}\, dk + \\ & \sum_{n=1}^{2} \left\{ \frac{2}{\pi} \int_0^{\theta_n^*} \int_0^{\infty} F_n^{(1)}(k,\theta)\, dk\, d\theta + \frac{2}{\pi} \int_{\theta_n^*}^{\frac{\pi}{2}} \int_{L_{1,n}} F_n^{(1)}(k,\theta)\, dk\, d\theta + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \int_{L_{2,n}} F_n^{(1)}(k,\theta)\, dk\, d\theta \right\} \end{aligned} \tag{21}$$

$$\begin{aligned} G^{(2)} &= -\int_0^{\infty} \frac{2\gamma \cosh(k(z+h_2)) \sinh(k(\zeta - h_1)) J_0(kR)}{\cosh kh_1 \cosh kh_2 + \gamma \sinh kh_1 \sinh kh_2}\, dk + \\ & \sum_{n=1}^{2} \left\{ \frac{2}{\pi} \int_0^{\theta_n^*} \int_0^{\infty} F_n^{(2)}(k,\theta)\, dk\, d\theta + \frac{2}{\pi} \int_{\theta_n^*}^{\frac{\pi}{2}} \int_{L_{1,n}} F_n^{(2)}(k,\theta)\, dk\, d\theta + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \int_{L_{2,n}} F_n^{(2)}(k,\theta)\, dk\, d\theta \right\} \end{aligned} \tag{22}$$

where

$$R^2 = (x-\xi)^2 + (y-\eta)^2, \qquad r_n^2 = R^2 + (z - \zeta - 2nh_1)^2, \qquad r_{1n}^2 = R^2 + (z + \zeta - 2nh_1)^2,$$

$$F_n^{(m)}(k,\theta) = \frac{gk\omega_n^2 P_n^{(m)} e^{-ik(x-\xi)\cos\theta} \cos(k(y-\eta)\sin\theta)}{\sinh kh_1 (\cosh kh_1 \cosh kh_2 + \gamma \sinh kh_1 \sinh kh_2)(\omega_1^2 - \omega_2^2)(\omega_n^2 - (kU\cos\theta + \sigma)^2)}.$$

The functions $P_n^{(m)}$ depend on z and ζ and are given in [4]. The angle θ_n^* depends on the speed U and the encounter frequency σ. For a given σ, there exists a solution k_n^* to the following equation

$$\sigma = \omega_n(k_n^*) - k_n^* C_{g_n}(k_n^*), \tag{23}$$

where $C_{g_n} = \partial \omega_n(k)/\partial k$ is the group velocity for mode n. If we define $U_n^* = C_{g_n}(k_n^*)$, then

$$\theta_n^* = \begin{cases} 0 & \text{if} \quad U < U_n^*, \\ \cos^{-1}(U_n^*/U) & \text{if} \quad U > U_n^*. \end{cases} \tag{24}$$

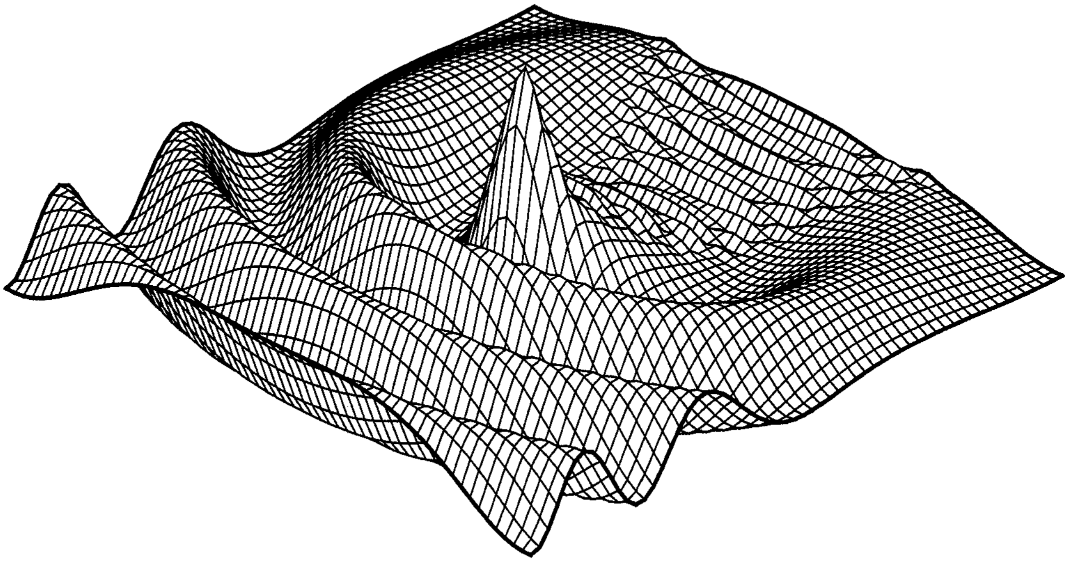

Figure 2: Internal waves due to point source, $\gamma = 0.9$, $h_1/h = 0.8$, $U/\sqrt{(gh)} = 0.05$, $\sigma\sqrt{(h/g)} = 0.187$

The paths $L_{1,n}$ and $L_{2,n}$ are defined in Fig. (1). The poles $k_{1,n}$ and $k_{2,n}$, with $k_{1,n} < k_{2,n}$, are the roots of $\omega_n^2 - (kU\cos\theta + \sigma)^2 = 0$ for $\theta_n^* \leq \theta \leq \pi/2$, and the poles $k_{3,n}$ and $k_{4,n}$, with $k_{3,n} < k_{4,n}$ are the roots for $\pi/2 \leq \theta \leq \pi$.

Fig. (2) shows the fluid interface elevation $\zeta^{(2)}$ due to the translating and pulsating source. $\zeta^{(2)}$ is obtained from $G^{(m)}$ as follows:

$$\zeta^{(2)} = \frac{1}{\epsilon g}(i\sigma(\gamma G^{(1)} - G^{(2)}) - U(\gamma G_x^{(1)} - G_x^{(2)})), \qquad z = 0. \tag{25}$$

Only the real component of $\zeta^{(2)}$ is shown in Fig. (2). This represents the internal waves at $t = 0$. Wave patterns due to the translation and oscillation of the source can be clearly seen. Surface and internal waves as well as hydrodynamic coefficients for a submerged spheroid traveling in the upper fluid layer will be presented at the workshop.

References

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