Hydroelastic Interaction of a Large Floating Platform with Head Seas

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1 Formulation

A thin plate of elongate form representing a floating airport covers a part of the undisturbed water surface coinciding with the $x-y$ plane of the coordinate system. The part of the $x-y$ plane covered by the airport is the region of $-B(x) \leq y \leq 0,\ 0 \leq x \leq L$ where $B(x)$ is the breadth at x and L the length of the plate. The positive z axis is taken to be vertically upward. Considering a conceptual design of the floating airport which is several kilometers long, several hundreds meters long and several meters thick, we may assume $L = +\infty$ and the thickness of the plate $d = 0$.

The airport will be built on the coastal zone where water depth is small compared with the length of sea waves; we use linear shallow water theory for the analysis of the hydroelastic interaction of water waves and the plate.

We suppose the regular waves being incident head on the plate whose velocity potential is written as

$$\phi_0(x,y)e^{i\omega t} = e^{-ikx+i\omega t} \tag{1}$$

We consider $B = \infty$ hereafter in our analysis because the width of the airport is very large compared with the wave length i.e. $kB >> 1$. One may readily extend the result with $B = \infty$ to the plate of very large but finite width.

Velocity potential $\phi(x,y)e^{i\omega t}$ and the wave elevation $\zeta(x,y)e^{i\omega t}$, which represents the vertical deflection of the plate in the '*Plate*' region of $-B(x) \leq y \leq 0,\ 0 \leq x \leq L$ and the water-wave elevation in the '*Water*' region other than '*Plate*' region, satisfy the kinematic condition

$$i\omega\zeta = -h\nabla^2\phi \tag{2}$$

in both the *Plate* and *Water* regions. Here

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

and h denotes the water depth.

In the *Water* region ϕ must satisfy the wave equation

$$\nabla^2\phi + k^2\phi = 0 \tag{3}$$

where the wave number k is given by

$$k = \omega/\sqrt{gh} \tag{4}$$

The following equation for ϕ in the *Plate* region derives from the equation of bending vibration of the plate

$$D\nabla^6\phi + \rho g\nabla^2\phi + \rho g k^2\phi = 0 \tag{5}$$

Here D is the bending rigidity of the plate per unit length, ρ density of the fluid and g the acceleration of gravity. We assume relatively small bending rigidity $D = O(k^{-4})$ in eq.(5) to retain the first term. Otherwise eq.(5) will lead to a different solution.

The dispersive relation corresponding to eq.(5), which determines the wave number k_* of the plate deflection in the region *Plate*, is

$$Dk_*^6 + \rho g k_*^2 - \rho g k^2 = 0 \tag{6}$$

which has two real roots $\pm k_\Lambda$.

Our problem is to solve for ϕ satisfying the plate equation (5) in the *Plate* and the wave equation (3) in the *Water* under the boundary conditions ensuring free bending moment and free shear force on the edge of the plate and the radiation condition in the far farfield.

Here we are concerned with the solution at the location $x = O(1)$ which is away form the corner at $(x = 0,\ y = 0)$. The plate is in the head sea of eq.(1) and it is reasonable to attempt a solution of the form

$$\phi(x, y) = \psi_1(x, y)e^{-ikx} + \psi_2(x, y)e^{-ik_\Lambda x} \tag{7}$$

where the variation of $\psi_{1,2}$ in x is small within a wave length, the characteristic scale being the plate length or breadth.

2 Solution for ψ_1

ψ_1 results mainly from the interaction of the incident waves with the edge of the plate along the x axis . The vibration of the plate at $x = O(1)$ will be restricted mainly within the distance of wave lengths to the edge of the plate ($y = -O(k^{-1})$). Therefore

$$\frac{\partial \psi_1}{\partial y} = O(k\psi_1) = O(k_\Lambda \psi) \tag{8}$$

Substitute $\psi_1 e^{i\omega t}$ into the plate equation (5) to retain the lowest order terms of $O(k^2)$, we have

$$D\left(-k^2 + \frac{\partial^2}{\partial y^2}\right)^3 \psi_1 + \rho g\left(-k^2 + \frac{\partial^2}{\partial y^2}\right)\psi_1 + \rho g k^2 \psi_1 = 0 \tag{9}$$

A solution of eq.(9) that is finite at $y \to -\infty$ is written in the form

$$\psi_1 = \sum_{j=1}^{3} B_j(x)e^{\mu_j y} \tag{10}$$

where μ_j are roots of eq.(11) with non negative real part.

$$D(\mu_j^2 - k^2)^3 + \rho g(\mu_j^2 - k^2) + \rho g k^2 = 0 \tag{11}$$

In the *Water* region of $y > 0$ the slowly varying character of ψ_1 allows us to drop $\partial^2\psi_1/\partial x^2$ compared with $\partial^2\psi_1/\partial y^2$ in the wave equation (3) and leads to the parabolic approximation.

$$-2ik\frac{\partial \psi_1}{\partial x} + \frac{\partial^2 \psi_1}{\partial y^2} = 0 \tag{12}$$

This approximation is valid in the *Water* region close to the edge along the x axis of $y = O(k^{-1/2})$. Now we ignore the effect of the corner at the origin of the coordinate system $(x = 0, y = 0)$ and therefore we require the initial condition for eq.(12) of no disturbance to the incident waves at $x = 0$

$$\psi_1 = 1 \quad \text{at} \quad x = 0 \quad \text{for} \quad y > 0 \tag{13}$$

and the far-field condition

$$\psi_1 \to 1 \quad \text{for} \quad y \to +\infty \quad \text{for all} \quad x \tag{14}$$

Then a solution of eq.(12) is given by

$$\psi_1 = 1 - \frac{1-i}{2\sqrt{\pi k}} \int_0^x d\xi \frac{V_1(\xi)}{\sqrt{x-\xi}} e^{-iky^2/2(x-\xi)} \tag{15}$$

with an unknown function $V_1(x)$.

Matching conditions of the solution (10) in the *Plate* region and (15) in the *Water* region are imposed at $y = 0$: conditions of free shear force, free bending moment and continuity of mass flux and energy flux. Determination of the unknowns P_j and V_1 is rather straightforward and the description is omitted.

3 Solution for ψ_2

Before solving ψ_2 we consider a different problem for ϕ_* when the plate extends from $y = -\infty$ to $y = +\infty$ without the edge along the x axis. ϕ_* is a 2D solution independent of y and readily obtained (Ohkusu and Namba 1998). At $x = O(1)$ the evanescent part of ϕ_* dies out; no reflection comes from the aft end of the plate at $x = \infty$. Therefore we have

$$\phi_*(x) = A_0 e^{-ik_\Lambda x} \tag{18}$$

where A_0 is determined with the conditions of free bending moment etc. at the edge along y axis.

Our solution $\psi_2 e^{-k_\Lambda x}$ must obviously approach to $A_0 e^{-ik_\Lambda x}$ as $y \to -\infty$ where the edge effect diminishes because it is far away from the plate edge along the x axis. ψ_2 will be considered as the interaction of the deflection wave $A_0 e^{-ik_\Lambda x}$ of the plate with the plate edge along the x axis.

The *Plate* region is divided into two domains, Region 1 ($y = -O(k_\Lambda^{-1/2})$) and Region 2 ($y = -O(k_\Lambda^{-1})$). The plate equation (5) for ψ_2 will be approximated by dropping the $x-$derivatives relative to the $y-$derivatives whose magnitude is of different order in Region 1 and 2:

$$(\rho g + 3Dk_\Lambda^4)\left(\frac{\partial^2 \psi_2}{\partial y^2} - 2ik_\Lambda \frac{\partial \psi_2}{\partial x}\right) = 0 \quad \text{(Region1)} \tag{19}$$

$$D\frac{\partial^6 \psi_2}{\partial y^6} - 3Dk_\Lambda^2 \frac{\partial^4 \psi_2}{\partial y^4} + (\rho g + 3Dk_\Lambda^4)\frac{\partial^2 \psi_2}{\partial y^2} = 0 \quad \text{(Region2)} \tag{20}$$

A solution of the parabolic approximation (19) to be matched with A_0 at $y = -\infty$ will be given in the form

$$\psi_2 = A_0 - \frac{1-i}{2\sqrt{\pi k_\Lambda}} \int_0^x d\xi \frac{V_2(\xi)}{\sqrt{x-\xi}} e^{-ik_\Lambda y^2/2(x-\xi)} \tag{21}$$

It is straightforward to derive a general solution of eq.(20). It is written as

$$\psi_2 = \sum_{j=1}^{4} E_j e^{\sigma_j y} + \alpha y + \beta \tag{22}$$

where E_j and α, β are constants. $\sigma_j (j = 1, 2, 3, 4)$ are roots of the equation

$$D\sigma^4 - 3Dk_\Lambda^2 \sigma^2 + (\rho g + 3Dk_\Lambda^4) = 0 \tag{23}$$

In the *Water* region of $y = +O(k^{-1/2})$, the wave equation (3) will be approximated by

$$\frac{\partial^2 \psi_2}{\partial y^2} + (k^2 - k_\Lambda^2)\psi_2 = 0 \tag{24}$$

We choose the solution of eq.(24) as

$$\psi_2 = R\exp[-i\sqrt{k^2 - k_\Lambda^2}y] \tag{25}$$

because $k > k_\Lambda$ and our solution is to satisfy the radiation condition that no waves come from $y = +\infty$. R is an unknown constant.

Matching conditions (22) and (25) at $y = 0$ are

$$E_1 + E_2 + \beta - R = 0, \quad \sigma_1 E_1 + \sigma_2 E_2 + \alpha + i\sqrt{k^2 - k_\Lambda^2}R = 0 \tag{26}$$

where $E_3 = E_4 = 0$ is chosen because $\sigma_{3,4}$ are negative in real part. Equation (21) at $y = 0_-$ must match with eq.(22) at $y = -\infty$. It yields

$$\alpha + V_2(x) = 0, \quad \beta + \frac{1-i}{2\sqrt{\pi k_\Lambda}} \int_0^x d\xi \frac{V_2(\xi)}{\sqrt{x - \xi}} = A_0 \tag{27}$$

Two other conditions are free shear force and free bending moment conditions at $y = 0$ which are

$$\sum_{j=1}^{2} \sigma_j P_j E_j + k_\Lambda^4 (2 - \nu)\alpha = 0, \quad \sum_{j=1}^{2} P_j E_j + k_\Lambda^4 \nu \beta = 0 \tag{28}$$

where ν is Poisson's ratio of the plate and

$$P_j = [(\sigma_j^2 - k_\Lambda^2)^2 - (1 - \nu)k_\Lambda^2(\sigma_j^2 - k_\Lambda^2)] \tag{29}$$

Six unknowns $E_{1,2}, \alpha, \beta, R,$ and V_2 are determined from six equations (26), (27) and (28). For example V_2 when $B(x)$ is constant is given by

$$V_2(x) = \frac{A_0}{Z} e^{-ix/2k_\Lambda Z^2} \operatorname{erfc}\Big(\frac{1-i}{2\sqrt{k_\Lambda}Z}\sqrt{x}\Big) \tag{30}$$

where erfc is the complementary error function and Z is a constant determined by P_j and σ_j

Once $V_2(x)$ is known, all the unknowns $(E_1, E_2, R, \alpha, \beta)$ are readily determined to give the deflection of the plate in Region 1,2 and the wave elevation in the *Water* region. We notice that it does not need any numerical computation to solve the problem except for evaluating the final integral expression for the plate deflection.

For the finite B the symmetrical solution from another edge at $y = -B$ is superposed to give the solution when $x = O(1)$. The plate deflection predicted by the present method agrees almost perfectly with the results by more computationally involved method i.e.a numerical boundary integral method using several thousands panels. Comparison of the results will be presented at the Workshop.

4 Reference

Ohkusu, M. and Namba, Y.: Hydroelastic Behavior of Floating Artificial Islands, vol.183, J.Japan Society of Naval Architects, pp239-248 (1998)