IMPULSIVE TSUNAMI GREEN FUNCTIONS FOR TWO-DIMENSIONAL BASINS

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1 Introduction

The word "tsunami" is increasingly being used as a technical term for any transient water wave that is generated by rapid normal deflections at the bottom. Such deformations may be due to earthquakes, landslides or falling rock. The physical mechanisms are the same for waves being generated in lakes, fjords, bays, harbours or the open sea.

It is now realized that the generation and the later propagation of tsunamis are two distinct physical processes. Hydrostatic long-wave theory may give a first approximation for a propagating tsunami, but it cannot give any adequate description of the generation process. This is obvious as we consider the Green function for a concentrated flux in a point at the bottom: The flux will obviously not produce a singular column of upwelling fluid. The impulsive upwelling will spread out smoothly horizontally. The local submergence depth for the source will manifest itself as the typical horizontal length scale for the upwelling region.

The present note considers the asymptotic limit of very rapid tsunami generation according to incompressible inviscid flow theory with small amplitude. The flow will be irrotational due to Kelvin's theorem. In this linearized theory, a concentrated impulsive bottom flux will produce a surface response satisfying the zero-potential condition at the surface. The actual bottom flux in a practical problem will occur along a finite length at the bottom. It can be built up by performing an integration over the Green function for a local concentrated flux. The velocity potential due to an arbitrary concentrated flux is the impulsive Green function for a given boundary geometry. The present note is devoted to constructing these Green functions and particularly their induced free-surface deflections for basic geometric configurations in two dimensions. Solutions may be constructed by the image method and conformal mapping techniques. In this note we will show in detail the case of an infinite sloping beach, plus some elementary solutions for a rectangular basin and a semi-circular basin.

2 Formulation

Consider a finite or semi-infinite body of fluid in two dimensions. The complex variable is $z = x + iy$. The bottom is described by the curve $z = z(s)$ where s is a real arc length coordinate along the bottom. The waterline is given by $s = 0$. The impulsive Green function $\phi(x, y; \xi)$ for the given geometry satisfies Laplace's equation for the velocity potential in the fluid domain apart from the singular point $z(\xi)$ at the boundary. The boundary is impermeable for all values of s except for $s = \xi$. Along the free surface $y = 0$ we have the equipotential condition $\phi = 0$. It follows from integrating Bernoulli's equation over the infinitesimal time of impulsive start, assuming constant pressure at the horizontal free surface. If n is a normal coordinate at the boundary (positive outwards), then the bottom condition is:

$$\frac{\partial \phi}{\partial n} = -\ \delta(s - \xi) \tag{1}$$

Dirac's delta function is denoted by δ. Excluding corner singularities, we assume the contour to be smooth at the point of concentrated flux $z = z(\xi)$. Then the singularity will constitute a sector of angle π of an isotropic source. So the dominating contribution to the Green function very close to the singular point will be:

$$\Phi(z; \xi) = \pi^{-1}\ \log[z - z(\xi)] \tag{2}$$

This source potential has strength equal to two, since it releases fluid on one side only. It is the complex potential, so we define $\phi = Re(\Phi)$. In the present note we will focus on the impulsive surface elevation defined by:

$$\eta(x; \xi) = \frac{\partial \phi(x, y; \xi)}{\partial y}, \quad y = 0 \tag{3}$$

3 Some basic solutions

The case of uniform depth has no waterline, so the definition of $s = 0$ is arbitrary. The singularity is taken in the point $(x, y) = (\xi, -1)$ at the bottom. The basic solution for constant unit depth is given by (see e.g. Tyvand 1998):

$$\eta(x;\xi) = \frac{1}{2}\, sech\left(\pi \frac{x-\xi}{2}\right) \tag{4}$$

By the method of images one can easily extend this solution to finite rectangular basins. We will here mention the simplest case where we introduce a rigid vertical wall at $x = 0$: The total surface elevation will then be:

$$\eta(x;\xi) = \frac{1}{2}\left[sech\left(\pi \frac{x-\xi}{2}\right) + sech\left(\pi \frac{x+\xi}{2}\right)\right] \tag{5}$$

We determine the value $\xi = 0.5611$, where there is an inflection point at $x = 0$. Only if the source is closer than this to the wall, the maximum surface elevation will occur at $x = 0$. Otherwise the maximum elevation will be located between the source location and the wall.

Another elementary example is the semi-circular basin. It is simply constructed by one source plus one image sink, and all streamlines are circles. Let the basin be given by the unit circle. In conventional polar coordinates (r, θ) the bottom will be described by $r = 1$ with $\pi < \theta < 2\pi$, so we define $s = \theta - \pi$. The surface elevation for a source at $s = \xi$ is:

$$\eta(x;\xi) = \frac{2}{\pi} \frac{\sin\xi}{((x+\cos\xi)^2 + \sin^2\xi)} \tag{6}$$

For $\xi = \pi/2$ this is the same formula as for a submerged impulsive source (Tyvand 1992, eq.21, taking $q_0 = 2$). The maximum elevation always occurs just above the source.

A more complicated case is the beach with constant slope angle α. We introduce a set of curvilinear coordinates (U, V):

$$U(x,y) + i\; V(x,y) = (x + \; i\; y)^{\frac{\pi}{2\alpha}} \tag{7}$$

The slope angle satisfies $0 < \alpha \leq \pi$. The U coordinate is taken as zero along the slope $arg(z) = -\alpha$ as well as the image slope $arg(z) = \alpha$. The V coordinate is taken as zero along the free surface. There is no physical length scale in the problem. So for simplicity the distance from the waterline to the singular point is taken as unity ($\xi = 1$). The isotropic source (with total flux 2) is then located in the point $z = exp(-i\alpha)$ in the complex plane, which will be represented as $(U, V) = (0, -1)$ in the curvilinear system. The image sink is located in the point $z = exp(i\alpha)$ which is the same as $(U, V) = (0, 1)$ in curvilinear coordinates. The tsunami Green function in this case is the velocity potential:

$$\phi = (2\pi)^{-1} \left[\log(U^2 + (V+1)^2) - \log(U^2 + (V-1)^2)\right] \tag{8}$$

By virtue of the definition of the curvilinear coordinates, we thus perform a conformal mapping from the (U, V) plane to the (x, y) plane. The mapping is here expressed in real variables. The free surface at $y = 0$ is mapped over to $V = 0$. The surface elevation is represented by the normal derivative at the free surface:

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0} = \left(\left.\frac{\partial \phi}{\partial V}\right|_{V=0}\right)\left(\left.\frac{\partial V}{\partial y}\right|_{y=0}\right) \tag{9}$$

We have:

$$\left.\frac{\partial \phi}{\partial V}\right|_{V=0} = \frac{2}{\pi(1+U^2)} \tag{10}$$

$$\left.\frac{\partial V}{\partial y}\right|_{y=0} = \frac{\pi}{2\alpha}\, x^{\left(\frac{\pi}{2\alpha}-1\right)} \tag{11}$$

In the final result we reintroduce an arbitrary singularity position ξ :

$$\eta(x;\xi) = \alpha^{-1}\left(1 + \left(\frac{x}{\xi}\right)^{\frac{\pi}{\alpha}}\right)^{-1}\left(\frac{x}{\xi}\right)^{\frac{\pi}{2\alpha}-1} \tag{12}$$

We note that the surface elevation at the waterline $x = 0$ is singular for $\alpha > \frac{\pi}{2}$.

References

[1] P. A. Tyvand, 1992, "Unsteady free-surface flow due to a line source", Phys. Fluids **A4**, 671-676.

[2] P. A. Tyvand, 1998, "Impulsive free-surface flow due to a line source at a bottom", manuscript.