THE OSCILLATING SUBMERGED SPHERE BETWEEN PARALLEL WALLS : THE CONVERGENCE OF THE MULTIPOLE EXPANSION

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1 Introduction

The following problem was brought to my attention by G.X.Wu who has been concerned with the computation of wave forces on bodies in a canal. The simplest case is a submerged sphere. Let rectangular cartesian axes (x, y, z) be taken with the origin in the mean free surface $y = 0$, where y increases with depth. A submerged sphere of radius a is placed with its centre at $(0, f, 0)$, midway between parallel vertical walls $x = \pm\ell$, where $\ell > a$ and $f > a$. The motion is assumed to have angular frequency ω and small amplitude. The normal velocity on the sphere is prescribed and is symmetrical about the mid-plane $x = 0$, antisymmetrical velocities can be treated similarly. (In fact the method is applicable for any submerged position of the sphere.) The corresponding velocity potential is denoted by $\phi(x, y, z)\exp(-i\omega t)$ and is to be found. (The time-factor $\exp(-i\omega t)$ will henceforth be omitted.) Then the governing equation is Laplace's equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi(x,y,z)=0, \tag{1.1}$$

with the boundary conditions

$$\frac{\partial\phi}{\partial x}=0 \text{ on } x=\pm\ell, \tag{1.2}$$

and the boundary condition

$$K\phi+\frac{\partial\phi}{\partial y}=0 \text{ on } y=0, \tag{1.3}$$

where $K = \omega^2/g$. Let spherical polar coordinates (r, θ, α) be taken about the centre of the sphere, such that

$$x = r\sin\theta\sin\alpha,\ y = f + r\cos\theta,\ z = r\sin\theta\cos\alpha, \tag{1.4}$$

where

$$r^2 = x^2 + (y-f)^2 + z^2.$$

Then the boundary condition on the sphere $r = a$ is of the form

$$\frac{\partial\phi}{\partial r}=U_0(\theta,\alpha)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}U(m,n)\left(\frac{(n-m)!}{(n+m)!}\right)^{1/2}\mathrm{P}_n^m(\cos\theta)\cos m\alpha, \tag{1.5}$$

where $U_0(\theta, \alpha)$ is a prescribed function with known coefficients $U(m, n)$ and is even in α. (As has been noted, odd functions can be treated similarly.) There is also a radiation condition at infinity: the waves travel outwards towards $z = \pm\infty$.

Here a brief account will be given of work which is to be published elsewhere, see [Ursell 1999]. The solution $\phi(x, y, z)$ of our boundary-value problem is written as the sum of multipole potentials,

$$\phi(x,y,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{m=n}C(m,n)\,a^{n+1}\left(\frac{(n-m)!}{(n+m)!}\right)^{1/2}(G_n^m)_\ell, \tag{1.6}$$

where the multipole potential $(G_n^m)_\ell$ has the singularity $r^{-n-1}\ \mathrm{P}_n^m(\cos\theta)\ \cos m\alpha$ at the centre of the sphere and satisfies the boundary conditions on the free surface, on the side walls, and at infinity. Expressions have been found for the multipole potentials, by methods which are complicated but essentially straightforward and which are briefly described in section 2 below. (These expressions are not unique, although the multipole potentials are uniquely defined.) When these potentials are expanded in spherical polar coordinates (1.4) and the boundary condition (1.5) is applied, the coefficients $C(m,n)$ in (1.6) are found to satisfy the system

$$C(m,n)+\sum_{N=0}^{\infty} a(m,n;N)C(m,N)+\sum_{N=0}^{\infty}\sum_{M=0}^{N} b(m,n;M,N)C(M,N)=-\frac{a}{n+1}\,U(m,n)=d(m,n),$$

$$0\le n<\infty,\ 0\le m\le n. \qquad (1.7)$$

The coefficients in the expansions involve single and double integrals. To complete the mathematical treatment it is necessary to examine whether the system (1.7) has a solution, and whether the resulting series (1.6) is convergent in the whole physical domain. Here it will be shown that for our problem our construction does indeed provide a valid solution, except possibly for a discrete set of values of K. This result is of interest because such convergence arguments have previously been given in only a few simple cases. The proof depends on the theory of infinite linear systems which is analogous to the theory of integral equations but simpler. We shall apply the following fundamental result :

Suppose that in the infinite system (1.7) the coefficients $a(m,n;N)$, $b(m,n;M,N)$ and $d(m,n)$ depend analytically on a parameter K and satisfy the conditions

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{N=m}^{\infty}|a(m,n;N)|^2<\infty \quad \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{M=0}^{\infty}\sum_{N=M}^{\infty}|b(m,n;M,N)|^2<\infty, \quad \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}|d(m,n)|^2<\infty.$$

Then there exists a unique solution $\{C(m,n)\}$ such that $\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}|C(m,n)|^2<\infty$, except possibly for a discrete set of values of K. The proof of this fundamental result is omitted, it is a simple adaptation of the classical proof when the terms $a(m,n;N)$ are absent, and involves only Schwarz's Inequality $(\sum|XY|)^2\le(\sum|X|^2)(\sum|Y|^2)$. It is easy to deduce from this result a stronger form of the theory: Suppose that in the infinite system (1.7) the coefficients $a(m,n;N)$, $b(m,n;M,N)$ and $d(m,n)$ satisfy the conditions

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{N=m}^{\infty}|a(m,n;N)|<\infty \quad \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{M=0}^{\infty}\sum_{N=M}^{\infty}|b(m,n;M,N)|<\infty, \quad \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}|d(m,n)|<\infty.$$

Then there exists a unique solution $\{C(m,n)\}$ such that $\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}|C(m,n)|<\infty$, except possibly for a discrete set of values of K. We shall see that we must prove the convergence of fourfold sums of single or double integrals.

In our derivation of the multipole potentials use is made of the ideas of Havelock's wavemaker theory [Havelock 1929] which is based on the following inversion theorem (the Havelock transform). Suppose that the function $f(y)$ is defined in the interval $0<y<\infty$, and suppose that the constant K is positive. Then the function $f(y)$ can be expanded in the form

$$f(y)=A_0e^{-Ky}+\int_0^{\infty}B_0(k)(k\cos ky-K\sin ky)dk, \qquad (1.8)$$

where

$$A_0=2K\int_0^{\infty}f(y')e^{-Ky'}dy' \text{ and } B_0(k)=\frac{2}{\pi(k^2+K^2)}\int_0^{\infty}f(y')(k\cos ky'-K\sin ky')dy'. \qquad (1.9)$$

This expansion is appropriate for functions satisfying the end condition

$$Kf(y)+\frac{d}{dy}f(y)=0 \text{ when } y=0.$$

2 The Multipoles

In our work the multipole potential $(G_n^m)_\ell$ in the expression (1.6) is constructed in two stages. In the first stage an expression $(G_n^m)_\infty$ was found for the multipole in the absence of side walls:

$$(G_n^m)_\infty = \frac{\mathrm{P}_n^m(\cos\theta)}{r^{n+1}}\cos m\alpha + \frac{(-1)^n}{(n-m)!}\int_0^\infty \frac{k+K}{k-K}\, k^n\, e^{-k(y+f)} J_m(k\rho)\, dk \cos m\alpha, \qquad (2.1)$$

where (here and later) the path of integration passes below the pole $k = K$. Then the radiation condition at infinity is satisfied . The potential $(G_n^m)_\infty$ can be transformed into a three-dimensional Havelock wavemaker expansion when $x > 0$, and similarly when $x < 0$. The normal velocities induced by $(G_n^m)_\infty$ on the side walls $x = \pm\ell$ can therefore be found explicitly as the sum of a single and a double integral. In the second stage of the construction these velocities are reversed, they generate a wave motion $(G_n^m)_{image}$ in the canal which can be found explicitly by use of the three-dimensional form of Havelock's wavemaker theory as the sum of a single and a double integral. Finally we write

$$(G_n^m)_\ell = (G_n^m)_\infty + (G_n^m)_{image}.$$

We quote the expressions for $(G_n^m)_{image}$:

$$\begin{aligned}(G_n^m)_{image} = {} & \frac{2}{(n-m)!}(-1)^{n+m}\, i^m\, K^{n+1} e^{-K(y+f)} \times \\ & \times \int_{-\infty-\frac{1}{2}\pi i}^{\infty+\frac{1}{2}\pi i} dw\, A_n^m(w)\exp(-Kz\sinh w)\cos(Kx\cosh w)\cosh m(w+\tfrac{1}{2}\pi i)\end{aligned} \qquad (2.2)$$

$$\begin{aligned}+ {} & \frac{4}{\pi(n-m)!} i^{n+m}\int_0^\infty dk\, \frac{k^n}{k^2+K^2}(k\cos ky - K\sin ky)F(k,K,f,n-m)\times \\ & \times \int_{-\infty}^{\infty} dv\, B_n^m(k,v)\exp(-ikz\sinh v)\cosh(kx\cosh v)\cosh m(v+\tfrac{1}{2}\pi i),\end{aligned} \qquad (2.3)$$

where

$$F(k,K,f,s) = \begin{cases} (k\cos kf - K\sin kf) & \text{when } s \text{ is an even integer,} \\ -i(k\sin kf + K\cos kf) & \text{when } s \text{ is an odd integer.}\end{cases} \qquad (2.4)$$

The factors involving y show (as has already been noted) that this expansion for $(G_n^m)_{image}$ is a wavemaker expansion of Havelock type. It is found that

$$A_n^m(w) = i\frac{\exp(iK\ell\cosh w)}{\sin(K\ell\cosh w)} \quad\text{and}\quad B_n^m(k,v) = \frac{\exp(-k\ell\cosh v)}{\sinh(k\ell\cosh v)}; \qquad (2.5)$$

these values are independent of m and n .

We next expand the integrals in terms of polar coordinates (1.4) and can then impose the boundary condition (1.5). We thus obtain expressions for the coefficients $a(m,n;N)$ and $b(m,n;M,N)$ in the system (1.7) in a form involving single and double integrals. The coefficients $a(m,n;N)$ come from the component $(G_n^m)_\infty$. Thus the coefficient of $r^N \mathrm{P}_N^M(\cos\theta)\cos M\alpha$ in $(G_n^m)_\infty$ is found to be

$$A(m,n,N) = \frac{(-1)^{n+N}}{(n-m)!(N+M)!}\int_0^\infty \frac{k+K}{k-K}\, k^{n+N}\, e^{-2kf}\, dk, \qquad (2.6)$$

and the coefficient of $r^N \mathrm{P}_N^M(\cos\theta)\cos M\alpha$ in $(G_n^m)_{image}$ is found to be

$$\begin{aligned}B(m,n,M,N) = {} & \epsilon_M \frac{K^N}{(N+M)!}\, e^{-2Kf}\frac{2}{(n-m)!}(-1)^{n+m+N+M}\, K^{n+1}\times \\ & \times \int_{-\infty-\frac{1}{2}\pi i}^{\infty+\frac{1}{2}\pi i} \frac{\exp(-K\ell\cosh w)}{\sin(K\ell\cosh w)}\cos m(\tfrac{1}{2}\pi - iw)\,\cos M(\tfrac{1}{2}\pi - iw)\, dw\end{aligned}$$

$$+ \quad \epsilon_M \frac{(-1)^M}{(N+M)!} \frac{4}{\pi(n-m)!} i^{n+m+N} \times$$

$$\times \int_0^\infty dk \frac{k^{n+N}}{k^2+K^2} F(k,K,f,n-m)\, F(k,K,f,N-M) \,.$$

$$\int_{-\infty}^\infty dv \, \frac{\exp(-k\ell\cosh v)}{\sinh(k\ell\cosh v)} \cosh m(v+\frac{1}{2}\pi i) \, \cosh M(v+\frac{1}{2}\pi i), \quad (2.7)$$

where $\epsilon_0 = 1$ and $\epsilon_M = 2$ when $M \geq 1$. It is then not difficult to show that

$$a(m,n;N) = -\frac{n}{n+1}\left(\frac{(n+m)!}{(n-m)!}\right)^{1/2} a^{N+n+1}\left(\frac{(N-m)!}{(N+m)!}\right)^{1/2} A(m,N,n), \quad (2.8)$$

and that

$$b(m,n;M,N) = -\frac{n}{n+1}\left(\frac{(n+m)!}{(n-m)!}\right)^{1/2} a^{N+n+1}\left(\frac{(N-M)!}{(N+M)!}\right)^{1/2} B(M,N,m,n). \quad (2.9)$$

If we can now find bounds for the integrals in (2.6) and (2.7) then we shall have bounds for $a(m,n;N)$ and $b(m,n;M,N)$. It has been found that bounds of the simple form

$$\left|\int f(X)\,dX\right| \leq \int |f(X)|\,|dX|$$

are sufficient for this purpose. We therefore have to show the convergence of series like

$$\sum_{N=0}^{\infty}\sum_{n=0}^{\infty}\sum_{M=0}^{N}\sum_{m=0}^{n} |b^{(2)}(m,n;M,N)| \,,$$

where

$$|b^{(2)}(m,n;M,N)| < A\left(\frac{a}{\ell}\right)^{N+n+1} \frac{\Gamma\{\frac{1}{2}(N+n+M+m+1)\}\Gamma\{\frac{1}{2}(N+n-M-m+1)\}}{\{(N+M)!(N-M)!(n+m)!(n-m)!\}^{1/2}}, \quad (2.10)$$

and this can be shown provided that $a < \ell$. Similarly we can show the validity of the multipole expansion in the whole field of flow, except that the components $(G_n^m)_\infty$ need a more careful treatment which was in effect given at the last Workshop in Holland, see [Ursell 1997].

3 Discussion

The foregoing treatment has been based on the wave potentials (2.1) and a Havelock wavemaker calculation. (It is not difficult to show that the method can be extended to any submerged position of the sphere and to arbitrary prescribed velocities on the sphere.) This seems to me the most natural approach. Wu in his work started from a different expression for the wave-source potential $(G_0^0)_\infty$. He then obtained the wave-source multipole by repeated reflection in the side walls and deduced a complete set of multipoles by differentiation. His approach and mine are equally valid, but I think that it would be difficult to use his expressions to prove convergence of his process. Wu's work, also containing his computations, was published in [Wu 1998]. Wu has recently checked my mathematical calculations and has used them to compute numerical values. He has found that these agree with his earlier computations in [Wu 1998].

References

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