THE USE OF THE CL-EQUATION AS A MODEL FOR SECONDARY CIRCULATIONS

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1. Introduction. In problems of interaction of waves and currents, it is customary that the influence of the current on the waves gets most attention. However, waves also influence the current itself. The most general and precise way to formulate this is via the socalled generalised Langrangian mean (GLM) method, introduced by Andrews and McIntyre (1978a,b); for an introduction we refer to McIntyre (1980) and §2.10.6 of Dingemans (1997). As shown by Leibovich (1980), see also Radder (1994) and Dingemans et al. (1996), the mean-current equation in GLM coordinates simplifies under mild conditions to the Craik-Leibovich equation in Eulerian coordinates (also denoted as CL equation), which reads:

$$\partial_t \overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}} \cdot \text{grad})\, \overline{\boldsymbol{u}} + \text{grad}\, \overline{\pi} = \overline{\boldsymbol{u}}^S \wedge \overline{\boldsymbol{\varpi}} + \rho^{-1}\, \text{div}\, \overline{\boldsymbol{\sigma}}' \ , \tag{1.1}$$

where $\text{div}\, \overline{\boldsymbol{\sigma}}' \equiv \partial \overline{\sigma}'_{ik}/\partial x_k$, the pressure term $\overline{\pi}$ is given by $\overline{\pi} = \overline{p}/\rho + gz + \frac{1}{2}\langle \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}} \rangle$, $\overline{\boldsymbol{u}}$ is the (Eulerian) mean velocity, $\overline{\boldsymbol{u}}^S$ is the Stokes drift, defined as the difference between the Lagrangian and Eulerian mean velocity, and $\tilde{\boldsymbol{u}}$ is the wave part of the velocity (the total velocity is considered the sum of the current and the wave part and the Stokes part of the velocity: $\boldsymbol{u} = \overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}} + \overline{\boldsymbol{u}}^S$). Notice that we have $\overline{\boldsymbol{u}}^S = \left(\overline{u}^S, \overline{v}^S, 0\right)^T$, $\overline{\boldsymbol{u}} = (\overline{u}, \overline{v}, \overline{w})^T$ and $\overline{\boldsymbol{\varpi}} = \text{curl}\, \overline{\boldsymbol{u}}$. It has been shown that Langmuir circulations can be generated through an instability mechanism of this CL equation; for a review is referred to Leibovich (1983). Essential for this to happen is the existence of the Stokes drift and the shear of the mean current, i.e., the vortex force $\overline{\boldsymbol{u}}^S \wedge \overline{\boldsymbol{\varpi}}$ is instrumental in the generation of these Langmuir vortex rolls. Because the Stokes drift is a wave-related quantity, it can be argued that one of the effects of waves on currents is the generation of Langmuir circulations.

Although no viscosity is taken into account in the usual CL-equation-formulations, it is advantageous to do so. This has to do with the so-called Large Eddy Simulation (LES) programs. Viscosity in these equations is needed for obtaining shear in the mean-current equations, which, in its turn, is needed to generate the vortex-force term. We take the (eddy) viscosity coefficient to be isotropic because of the scales on which the flow occurs here. Applying the Boussinesq-hypothesis, the stresses $\overline{\sigma}'_{ik}$ are approximated as[1] $\rho^{-1}\overline{\sigma}'_{ik} = \nu_T\, (\partial \overline{u}_i/\partial x_k + \partial \overline{u}_k/\partial x_i)$, while the eddy viscosity ν_T has still to be determined.

2. The equations for primary and secondary flow. One of the explanations of Langmuir circulations rests upon the supposition that an instability mechanism in the CL equation is responsible for the generation of these vortex rolls. We suppose that the mean current $\overline{\boldsymbol{u}}$ is disturbed. These perturbations are supposed to be of periodic nature, i.e., $\hat{\boldsymbol{u}}$ obeys a WKBJ-type of behaviour which is natural, also in view of the resulting (periodic) vortex-roll motions. We then have the situation that $\overline{\boldsymbol{u}}$ can be written as $\overline{\boldsymbol{u}} = \boldsymbol{U} + \hat{\boldsymbol{u}}$, where $\hat{\boldsymbol{u}}$ is the disturbance which is responsible for the formation of the vortex rolls and $\boldsymbol{U}$ is the velocity of the basic state. For the other quantities $\overline{\pi}$ and $\overline{\boldsymbol{\varpi}}$ in the CL equation the same kind of perturbations are assumed to exist, viz. a basic state (denoted with captitals) and a perturbed state (denoted by hatted variables).

Because the eddy viscosity is a function of the velocity $\overline{\boldsymbol{u}}$ and the depth z, a perturbation of ν_T is also necessary. It is shown by Dingemans (1999) that the perurbation of ν_T has no effect on the present results, for the order considered. We now simply write $\overline{\nu}_T$ in order to stress the approximation.

We now insert the expressions $\overline{\boldsymbol{u}} = \boldsymbol{U} + \hat{\boldsymbol{u}}$, $\overline{\boldsymbol{\varpi}} = \boldsymbol{\Omega} + \hat{\boldsymbol{\omega}}$, $\overline{\boldsymbol{\sigma}}' = \boldsymbol{\Sigma}' + \hat{\boldsymbol{\sigma}}'$, $\overline{\boldsymbol{u}}^S = \boldsymbol{U}^S$ in the CL equation and the continuity equation. Notice that the wave-related quantities, $\overline{\boldsymbol{u}}^S$ and $\tilde{\boldsymbol{u}}$, are not perturbed. Because of the periodicity of the perturbed quantities, averaging over a period and length large to the characteristic period and length of the perturbations, yields for the basic state:

$$\partial_t \boldsymbol{U} + (\boldsymbol{U} \cdot \text{grad})\, \boldsymbol{U} + \langle (\hat{\boldsymbol{u}} \cdot \text{grad})\, \hat{\boldsymbol{u}} \rangle + \text{grad}\, \Pi = \boldsymbol{U}^S \wedge \boldsymbol{\Omega} + \rho^{-1}\, \text{div}\, \boldsymbol{\Sigma}' \ , \tag{2.1}$$

Notice that $\langle (\hat{\boldsymbol{u}} \cdot \text{grad})\, \hat{\boldsymbol{u}} \rangle$ are the Reynolds stresses. The evaluation of these stresses is the subject of this study. The equation for the perturbation is obtained by subtraction of (2.1) from the full equation:

$$\partial_t \hat{\boldsymbol{u}} + (\hat{\boldsymbol{u}} \cdot \text{grad})\, \boldsymbol{U} + (\boldsymbol{U} \cdot \text{grad})\, \hat{\boldsymbol{u}} + (\hat{\boldsymbol{u}} \cdot \text{grad})\, \hat{\boldsymbol{u}} - \langle (\hat{\boldsymbol{u}} \cdot \text{grad})\, \hat{\boldsymbol{u}} \rangle + \text{grad}\, \hat{\pi} = \boldsymbol{U}^S \wedge \hat{\boldsymbol{\omega}} + \rho^{-1}\, \text{div}\, \hat{\boldsymbol{\sigma}}' \ . \tag{2.2}$$

The continuity equation splits in one for the basic state and one for the perturbed velocity:

$$\text{div}\, \boldsymbol{U} = 0 \quad \text{and} \quad \text{div}\, \hat{\boldsymbol{u}} = 0 \ . \tag{2.3}$$

[1] We write σ'_{ik} with the prime, denoting the part of the stress tensor without the pressure, see Dingemans (1997, p. 4.)

3. Simplified equations. We now adopt a number of simplifications. Firstly, we suppose that the basic current $\boldsymbol{U}$ is uniform in the horizontal directions, $\boldsymbol{U} = \boldsymbol{U}(z,t)$ and, moreover, no vertical component exists: $\boldsymbol{U} = (U(z,t), V(z,t), 0)^T \equiv \boldsymbol{U}^h$. This means that (nearly) horizontal nearly-uniform shear flows are considered. As pointed out in Dingemans (1997, pp. 193 and 201), the vertical component of the mean current can only be neglected when the bottom is (nearly) horizontal. It is therefore also supposed that *the bottom is horizontal*, i.e. $\nabla h(x,y) = 0$ where $\nabla = (\partial_x, \partial_y)^T$. Secondly, the wave-induced perturbation $\hat{\boldsymbol{u}}$ is supposed to be single-periodic in one specific direction θ. Thirdly, with θ the angle between the positive $x-$axis and the path of propagation s and n the lateral direction, we also suppose that $\partial\hat{\boldsymbol{u}}(x,y,z,t)/\partial n = 0$. Fourthly, the Stokes drift $\boldsymbol{U}^S$ is supposed to be only a function of depth, i.e., $\boldsymbol{U}^S = \boldsymbol{U}^S(z) = \left(U^S(z), V^S(z), 0\right)^T$. As the eddy viscosity is also a function of space and time through its dependence on the friction velocity, we now suppose that $\overline{\nu}_T = \overline{\nu}_T\left(\left|\boldsymbol{U}^*\left(\boldsymbol{X},T\right)\right|, z\right)$ with $\boldsymbol{X} = \delta\boldsymbol{x}$ and $T = \delta t$ and $\delta \ll 1$. The simplified momentum equations then become (details in Dingemans, 1999):

$$\partial_t \boldsymbol{U}^h + \left\langle \partial_{x_j}\left(\hat{u}_j \hat{\boldsymbol{u}}\right)\right\rangle^h + \nabla\Pi_0 = \partial_z\left(\overline{\nu}_T \partial_z \boldsymbol{U}^h\right) \quad \text{where} \quad \Pi_0 = \overline{P}/\rho + \tfrac{1}{2}\left\langle \tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right\rangle \ . \tag{3.1}$$

We note that in the present approximation the vortex force has only a vertical component and therefore plays no role in the horizontal mean momentum equations. For the perturbed velocity we get:

$$\partial_t \hat{\boldsymbol{u}}^h + \hat{w}\partial_z \boldsymbol{U}^h + \left(\boldsymbol{U}^h\cdot\nabla\right)\hat{\boldsymbol{u}}^h + \nabla\hat{\pi} = \left(\boldsymbol{U}^S \wedge \hat{\boldsymbol{\omega}}\right)^h + \rho^{-1}\left(\operatorname{div}\hat{\boldsymbol{\sigma}}'\right)^h \tag{3.2a}$$

where the viscosity term is a function on $\hat{\boldsymbol{u}}$ and $\overline{\nu}_T$. The vertical momentum equation becomes:

$$\partial_t \hat{w} + \left(\boldsymbol{U}^h\cdot\nabla\right)\hat{w} + \partial_z\hat{\pi} = \left(U^S\hat{\omega}_2 - V^S\hat{\omega}_1\right) 2\partial_z\left(\overline{\nu}_T\left(\partial_z\hat{w}\right)\right) + \overline{\nu}_T\partial_{x_j}\left(\partial_{x_j}\hat{w} + \partial_z\hat{u}_j\right) \quad j = 1,2 \ . \tag{3.2b}$$

4. Linear stability analysis. A solution of Eqs. (3.2) is sought now. Following Cox (1997) an asymptotic solution is sought by applying a long-wave expansion. This expansion is based on the observation that Langmuir circulations have a much larger horizontal extent (in the direction perpendicular to the circulation) than the extent of the circulation cells. The boundary conditions for the perturbed velocities then are:

$$\partial_z\hat{u} = \partial_z\hat{v} = w = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad \partial_z\hat{u} = \partial_z\hat{v} = w = 0 \quad \text{at} \quad z = -h \ . \tag{4.1}$$

It is noted that the conditions (4.1) do not comply with the no-slip conditions, which should apply for viscous flow as is considered here. It seems reasonable to limit this stability analysis to the bulk of the fluid, just outside the bottom boundary layer. We now assume a slow growth rate σ and the expansions of $\hat{\boldsymbol{u}}$ is:

$$\hat{\boldsymbol{u}}(\boldsymbol{x},z,t) = \boldsymbol{u}'(z)\,\mathrm{e}^{\vartheta} \quad \text{with} \quad \boldsymbol{u}'(z) = \hat{\boldsymbol{u}}_0(z) + \varepsilon\hat{\boldsymbol{u}}_1(z) + \varepsilon^2\hat{\boldsymbol{u}}_2(z) + \cdots \quad \text{where} \tag{4.2a}$$

$$\vartheta(\boldsymbol{x},t) = i\varepsilon\tilde{\boldsymbol{k}}\cdot\boldsymbol{x} + \varepsilon\sigma t \quad \text{and} \quad \sigma = \sigma_1 + \varepsilon\sigma_2 + \cdots \tag{4.2b}$$

with $\tilde{\boldsymbol{k}} = \left(\tilde{k}_1, \tilde{k}_2\right)^T$ the scaled wave number vector: $\tilde{\boldsymbol{k}} = \boldsymbol{k}/\varepsilon$ and $\left|\tilde{\boldsymbol{k}}\right| = \mathcal{O}(1)$. For $\hat{\pi}$ we use a similar expansion. In this way one focusses attention to the most unstable wave numbers $\boldsymbol{k}$, which are $\mathcal{O}(\varepsilon)$.

The expansions (4.2) are substituted in the linearised momentum equations for the perturbed velocities (3.2a) and (3.2b). The continuity equation yields $\varepsilon i\tilde{\boldsymbol{k}}\cdot\boldsymbol{u}' + \partial w'/\partial z = 0$. In the zeroth-order equation the continuity equation yields $w_0 =$ constant, and from the boundary conditions it then follows that $w_0 \equiv 0$. The zeroth-order momentum equations then simplify to $\partial_z u_0 = \partial_z v_0 = 0$. Using the boundary conditions it follows that u_0 and v_0 are constant in the fluid domain. It then also follows that π_0 is constant.

In first-order the continuity equation is $\tilde{\boldsymbol{k}}\cdot\boldsymbol{u}_0 + \partial_z w = 0$ and the boundary conditions in first order are $\partial_z u_1 = \partial_z v_1 = w_1 = 0$ at $z = -h$ and $z = 0$. This results in $w_1 \equiv 0$. From the bottom condition we have the condition $\tilde{k}_1 u_0 + \tilde{k}_2 v_0 = 0$ which serves as a relation between the unknown constants u_0 and v_0. The horizontal first-order momentum equations are integrated over depth (from $z = -h$ to $z = 0$). We introducing vertically-averaged quantities, denoted by a double overbar by $\overline{\overline{\boldsymbol{U}}} = \left(\overline{\overline{U}}, \overline{\overline{V}}\right)^T = h^{-1}\int_{-h}^{0}\boldsymbol{U}(z)dz$ and similarly for $\overline{\overline{\boldsymbol{U}^S}}$. These vertically-averaged equations are solved for σ_1 and π_0. It is clear that σ_1 is imaginary, otherwise it had to be zero since $u_0 \neq 0$ and $v_0 \neq 0$. We therefore write $\sigma_1 = i\sigma_1^{(i)}$. We obtain as solutions $\pi_0 = \boldsymbol{u}_0\cdot\overline{\overline{\boldsymbol{U}^S}}$ and $\sigma_1^{(i)} = -\tilde{\boldsymbol{k}}\cdot\left(\overline{\overline{\boldsymbol{U}}} + \overline{\overline{\boldsymbol{U}^S}}\right)$.

In second order we proceed as follows. Differentiation of the second-order continuity equation yields $i\tilde{k}_1\partial_z\left(\overline{\nu}_T\partial_z u_1\right) + i\tilde{k}_2\partial_z\left(\overline{\nu}_T\partial_z v_1\right) + \partial_z\left(\overline{\nu}_T\partial_z^2 w_2\right) = 0$. Expressions for $\partial_z\left(\overline{\nu}_T\partial_z u_1\right)$ and $\partial_z\left(\overline{\nu}_T\partial_z v_1\right)$ follow from the (unaveraged) first-order equations. For w_2 we then obtain the differential equation:

$$\partial_z\left(\overline{\nu}_T\partial_z^2 w_2\right) = \left|\tilde{\boldsymbol{k}}\right|^2 \boldsymbol{u}_0\cdot\left(\overline{\overline{\boldsymbol{U}^S}} - \boldsymbol{U}^S\right) \ . \tag{4.3}$$

The right-hand side is thus zero when no shear is present (i.e., when $\boldsymbol{U}^S$ is constant over the depth). Recapitulating, we have the unknown constants u_0 and v_0 with relation $\tilde{k}_1 u_0 + \tilde{k}_2 v_0 = 0$ between them, and solutions for π_0 and σ_1. For the first non-zero vertical velocity component we have the differential equation (4.3). In next section we consider the energy equation for the perturbed veocities in order to close the system.

5. The Landau-Stuart equation. The energy equation for the perturbed velocities $\hat{\boldsymbol{u}}$ follows by scalar multiplication of the momentum equation for the perturbed velocities with $\hat{\boldsymbol{u}}$. Introducing the mean kinetic energy by $\mathcal{K} = \iiint dxdydz \frac{1}{2}\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}} \equiv \int_{-h}^{0} dz \left\langle \frac{1}{2}\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}} \right\rangle$, the total change in kinetic energy may be written down, e.g. Joseph (1976, pp. 11-12). Using the simplifications of §3, the result is:

$$(5.1) \quad \frac{d\mathcal{K}}{dt} \equiv \frac{d}{dt}\left\{ \int_{-h}^{0} dz \left\langle \frac{1}{2}\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}} \right\rangle \right\} = -\int_{-h}^{0} dz \left\{ \langle \hat{w}\hat{u}_j \rangle \frac{\partial}{\partial z}\left(U_j^h + U_j^S\right) \right\} - \int_{-h}^{0} dz \left\{ \overline{\nu}_T \left\langle \left(\frac{\partial \hat{u}_i}{\partial x_j} \right)^2 \right\rangle \right\} .$$

An amplitude $\mathsf{A}_0 = \sqrt{u_0^2 + v_0^2}$ is introduced and we also write $\varepsilon^2 w_2(z) = \varepsilon^2 \tilde{k}^2 m_2(z) = k^2 m_2(z)$ where $\tilde{k}^2 = \tilde{k}_1^2 + \tilde{k}_2^2$. Instead of expansion (4.2) we now have the expansion

$$(5.2) \qquad \hat{\boldsymbol{u}}(\boldsymbol{x}, z, t) = \frac{1}{2}\left(\tilde{k}_2/\tilde{k}, -\tilde{k}_1/\tilde{k}, \varepsilon^2 \tilde{k}^2 m_2(z) \right)^T \mathsf{A}_0 \, \mathrm{e}^{\vartheta} + CC$$

with $\vartheta = i\varepsilon \tilde{\boldsymbol{k}} \cdot \boldsymbol{x} + i\sigma_1^{(i)} t$. For m_2 we have the differential equation

$$(5.3) \qquad \partial_z \left(\overline{\nu}_T(z) \partial_z^2 m_2 \right) = \tilde{k}^{-1} \left[\tilde{k}_2 \left(\overline{\overline{U^S}} - U^S \right) - \tilde{k}_1 \left(\overline{\overline{V^S}} - V_S \right) \right] \equiv G ,$$

with the boundary conditions $m_2(z) = \partial_z^2 m_2(z) = 0$ at $z = -h$ and $z = 0$. Introducing the notation $q = \tilde{k}_1/\tilde{k}_2$, the solution for $m_2(z; q)$ may be written as

$$(5.4) \quad m_2(z; q) = \int_0^z d\hat{z} f(\hat{z}; q) + \frac{z}{h} \int_0^{-h} d\hat{z} f(\hat{z}; q) \quad \text{with} \quad f(\hat{z}; q) = \int_0^{\hat{z}} dz' \frac{1}{\overline{\nu}_T(z'} \int_{-h}^{z'} dz'' G\left(z''; q\right) .$$

We consider the simplified energy equation (5.1). Following Stuart (1958), we now suppose the amplitude A_0 to be a function of time, $\mathsf{A}_0 = \mathsf{A}_0(t)$. Using the expansion (5.2) in this energy equation leads to the so-called Landau-Stuart equation:

$$(5.5) \qquad \frac{d\mathsf{A}_0^2}{dt} = 2\alpha \mathsf{A}_0^2 - \ell \mathsf{A}_0^4 \quad \text{with exact solution} \quad \mathsf{A}_0^2 = 1 \Big/ \left[\frac{\ell}{2\alpha} + \left(\frac{1}{\mathsf{A}_0^2} - \frac{\ell}{2\alpha} \right) \mathrm{e}^{-2\alpha t} \right] ,$$

where the coefficients α and ℓ consist of expressions in m_2, the Stokes drift, etc., see Dingemans (1999). We have $\ell > 0$, but the sign of α is not clear beforehand. An exact solution is found by rewriting Eq. (5.5) in one for A_0^{-2}, which equation turns out to be linear. When $\alpha > 0$, the solution (5.5) approaches the equilibrium solution, $\mathsf{A}_0^2 \to \mathsf{A}_e^2 = 2\alpha/\ell$ for $t \to \infty$. When $\alpha < 0$, $\mathsf{A}_0 \to 0$ for $t \to \infty$.

6. The alignment of the vortex rolls. To obtain the direction of the axis of the vortex rolls we now use the *principle of exchange of stability* (PES). Some remarks on PES can be found in Joseph (1976, pp. 26,27 and 55). The method was originally proposed by Stuart (1958). We have investigated the stability of perturbations of the form $\hat{\boldsymbol{u}}(\boldsymbol{x}, z, t) = \boldsymbol{u}' \exp\left[\vartheta(\boldsymbol{x}, t)\right]$. Here is $\vartheta = i\tilde{\boldsymbol{k}} \cdot \boldsymbol{x} + \varepsilon\sigma t$, indicating that in horizontal space the solution is periodic and in time growth or decay of the solutions may occurr. It was found that only an imaginary part of σ resulted. When this part is unequal to zero, then neutrally-stable solutions exist. When the imaginary part is also zero for one or more of the solutions of σ, then a bifurcation of the basic flow into a secondary flow may result. This secondary flow may be stable or unstable, depending on the prevailing conditions. Because we look for the generation of secondary currents due to the instability of the basic current, PES may well be valid in our case. We have $\sigma_1^{(i)} = 0$ when $\tilde{\boldsymbol{k}}_c \cdot \left(\overline{\overline{\boldsymbol{U}}} + \overline{\overline{\boldsymbol{U}^S}} \right) = 0$, or, in terms of q, $q_c \left(\overline{\overline{U}} + \overline{\overline{U^S}} \right) + \left(\overline{\overline{V}} + \overline{\overline{V^S}} \right) = 0$.

The wave number vector $\tilde{\boldsymbol{k}}$ points in the direction with the smallest periodicity of the periodic structure. In the perpendicular direction the component is very small, signifying that the extent of the periodicity is very large. The axis of the vortex roll is thus in a direction perpendicular to $\tilde{\boldsymbol{k}}$. For the special case that both $\overline{\overline{\boldsymbol{U}}}$ and $\overline{\overline{\boldsymbol{U}^S}}$ are in the x-direction so that $\tilde{k}_2 = 0$, we have $\tilde{k}_1 = 0$, signifying infinitely long rolls in the x-direction.

7. Maximal growth of the perturbations and the Reynolds stresses. Instead of PES we use a different method to determine the critical direction given by $q_c = \tan \varphi_c$ for which maximum growth of the perturbations occurs. When considering infinitesimal perturbations, the Landau-Stuart equation can be linearised to give $\max_q d\mathsf{A}_0^2/dt = 2\alpha \mathsf{A}_0^2$. Maximum growth is obtained for $d\alpha/dq = 0$ together with the condition that $d^2\alpha/dq^2 < 0$. Using the expressions for the coefficients of the Landau-Stuart equation, the value of q_c can be determined, see Dingemans (1999).

The Reynolds stresses $\langle \hat{w}\hat{u} \rangle = \varepsilon^2 \langle w_2 \hat{u} \rangle$ and $\langle \hat{w}\hat{v} \rangle$ can now be calculated. Returning to unscaled variables we see that for the case that $\alpha > 0$ we have $\langle \hat{w}\hat{u} \rangle = \frac{1}{2} k^2 \mathsf{A}_0^2 m_2 / \sqrt{1+q^2}$ and $\langle \hat{w}\hat{v} \rangle = \frac{1}{2} k^2 \mathsf{A}_0^2 q m_2 / \sqrt{1+q^2}$. When $\alpha < 0$ the Reynolds stresses are zero. For the amplitude A_0 we now use the equilibrium solution $\mathsf{A}_e = \sqrt{2\alpha/\ell}$. From the resulting expressions (see Dingemans, 1999) it appears that, to leading order, the Reynolds stresses do not depend on $k = \varepsilon \tilde{k}$, meaning that they are independent of the extent of the circulation cells (the size being proportional to $1/k$).

8. An example. We consider an example from flume experiments by Klopman (1994). Measurements show the influence of waves on currents, see Figure 8.1. Taking the logarithmic velocity profile $U(z) = (\overline{u}^*/\kappa) \log \{(z+h)/z_0\}$ for $-h + z_0 \leq z \leq 0$, we have $\overline{u}^*/\kappa = 0.018$ with $z_0 = 0.4$ mm and $h = 0.5$ m. For this case an approximate calculation of the radiation stress, using long-wave approximations, yields a current contribution $u_w = -0.24 h \log(2 + z/h)$. Determination of the mean current to be the same in the no-waves and waves case yields a constant $c = 0.0464$ m/s. In Figure 8.1 is given the $U(z)$ and the curve for $U(z) + u_w(z) + c$. It is clear that the effect of a backwards leaning velocity profile for following waves is included in the present theory.

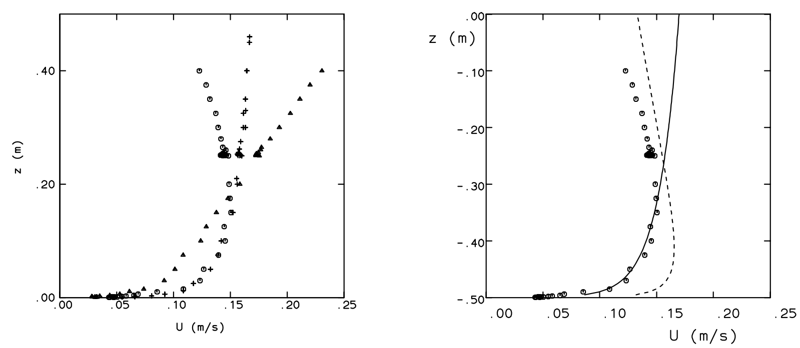

FIGURE 8.1. *Left: Klopman's (1994) measurements; +: current without waves, ∘: waves following the current, △: waves opposing the current. Right: drawn line: logarithmic current profile, interupted line: total velocity, circles: Klopman's measurements for following waves.*

REFERENCES

[1] **Andrews, D.G. and McIntyre, M.E., 1978a**. An exact theory of nonlinear waves on a Lagrangian mean flow. *J. Fluid Mechanics* **89(4)**, pp. 609-646.

[2] **Andrews, D.G. and McIntyre, M.E., 1978b**. On wave action and its relatives. *J. Fluid Mechanics* **89(4)**, pp. 647-664.

[3] **Cox, S.M., 1997**. Onset of Langmuir circulation when shear flow and Stokes drift are not parallel. *Fluid Dynamics Research* **19**, pp. 149-167.

[4] **Dingemans, M.W., 1997**. *Water Wave Propagation over Uneven Bottoms.* World Scientific, Singapore, 967 pp.

[5] **Dingemans, M.W., 1999**. 3D wave-current modelling; a model for secondary circulations. *WL|Delft Hydraulics Report* Z2612, 68 pp.

[6] **Dingemans, M.W., van Kester, J.A.Th.M., Radder, A.C. and Uittenbogaard, R.E., 1996**. The effect of the CL-vortex force in 3D wave-current interaction. *Proc. 25th Int. Conf. on Coastal Engineering, Orlando*, pp. 4821-4832.

[7] **Drazin, P.G. and Reid, W.H., 1981**. *Hydrodynamic Stability.* Cambridge University Press, 527 pp.

[8] **Joseph, D.D., 1976**. *Stability of Fluid Motion*, Vol. I. Springer Tracts in Natural Philosophy, Vol. 27, Springer Verlag, Berlin, 282 pp.

[9] **Klopman, G., 1994**. Vertical structure of the flow due to waves and currents. *Delft Hydraulics*, Report H840.30 Part II.

[10] **Leibovich, S., 1980**. On wave-current interaction theories of Langmuir circulations. *J. Fluid Mechanics* **99(4)**, pp. 715-724.

[11] **Leibovich, S., 1983**. The form and dynamics of Langmuir circulations. *Ann. Rev. of Fluid Mech.* **15**, pp. 391-427.

[12] **McIntyre, M.E., (1980)**. Towards a Lagrangian-mean description of stratospheric circulations and chemical transports. *Phil. Trans. Roy. Soc. London* **A296**, 129-148.

[13] **Radder, A.C., 1994**. A 3D wave-current interaction theory based on the CL equation. *Rijkswaterstaat/RIKZ* report RIKZ/OS-94.163x, Dec. 1994.

[14] **Rodi, W., 1980**. *Turbulence Models and their Application in Hydraulics.* IAHR, Delft, 104 pp.

[15] **Stuart, J.T., 1958**. On the non-linear mechanics of hydrodynamic stability. *J. Fluid Mech.* **4**, pp. 1-21.