A CONTINUUM MODEL FOR MULTI-COLUMN STRUCTURES IN WAVES

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At the last Workshop Phil McIver showed how ideas commonplace in solid-state physics, and in particular in lattice theory for crystals, could be used in connection with the problems of ocean wave interaction with periodic multi-column configurations such as might be used to support massive off-shore structures such as aircraft run-ways. During the discussion it was remarked that an alternative approach might be to look at homogenisation theory in which the (very large number of) columns were replaced by a continuous medium having some averaged different properties from the column-free medium, which describes in some global sense the effect of the columns. Just such an approach is presented here, drawing on a model which has been used with success in modelling the acoustic resonances which can occur in banks of heat exchangers in the form of a large number of periodic closely-spaced cylindrical tubes having axes at right-angles to the air flow.

To fix ideas we assume that the multi-column configuration occupies the region $|x| < a$, $0 < z < h$, i.e. extends indefinitely in the y-direction. The undisturbed free surface is $z = 0$ and the velocity of the fluid normal to the surface of each of the cylinders vanishes. It is thus possible to remove the vertical z-dependence by writing the three-dimensional harmonic velocity potential $\Phi(\boldsymbol{r}, t)$ in the form

$$\Phi(\boldsymbol{r}, t) = \Re\phi(x, y) \cosh k(h - z) e^{-iwt}. \tag{1}$$

Then $\phi(x, y)$ satisfies

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0, \qquad |x| > a, \quad -\infty < y < \infty. \tag{2}$$

Also, the linearised free surface condition is satisfied provided the wavenumber k is the real, positive root of

$$w^2 = gk \tanh kh \tag{3}$$

where $w/2\pi$ is the assumed harmonic wave frequency. Thus the reduced potential ϕ satisfies the Helmholtz equation in $|x| > a$, all y, and in $|x| < a$ external to the region of the $x - y$ plane occupied by the cross-sections of the columns. In addition

$$\phi_n = 0 \text{ on each of the column boundaries.} \tag{4}$$

Suitable conditions need to be satisfied by ϕ as $|x| \to \infty$, depending on the problem under consideration.

Now if $k = w/c_0$, where c_0 is the speed of sound, then equations (2), (4) also describe the two-dimensional small perturbations, restricted to the $x - y$ plane, in sound pressure of a compressible fluid having frequency $w/2\pi$ in the presence of an array of rigid cylinders in $|x| < a$, $-\infty < y < \infty$, $0 < z < h$, on which a 'hard' condition is satisfied. This follows since the linearised continuity equation

$$\frac{\partial \rho}{\partial t} + \rho_0 \boldsymbol{\nabla}.\boldsymbol{u} = Q \tag{5}$$

where ρ_0 is the mean and ρ the actual density of the fluid and Q is any mass flux per unit volume and the linearised momentum equation

$$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p = \boldsymbol{F} \tag{6}$$

where $\boldsymbol{F}$ is any external force per unit volume, reduce, if $Q = \boldsymbol{F} = 0$ to

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p. \tag{7}$$

Now assuming $p = p(\rho_0) + (\rho - \rho_0)p'(\rho_0)$ + smaller terms gives

$$\frac{\partial p}{\partial t} = p'(\rho_0)\frac{\partial \rho}{\partial t} \equiv c_0^2 \frac{\partial \rho}{\partial t}, \text{ say} \tag{8}$$

so that (7) becomes

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = \nabla^2 p \tag{9}$$

Finally, by letting

$$p(x, y, t) = \Re \phi(x, y)e^{-iwt} \tag{10}$$

we recover (2) with $k = w/c_0$.

Thus the water wave problem for the 'reduced' potential ϕ and the acoustic problem in the x, y plane are identical and we may adapt a homogenisation technique used successfully for the latter problem, and described by Blevins [1], to the former problem.

The aim is to replace the region $|x| < a$, $0 < z < h$ occupied by the array of cylinders, by a homogeneous medium satisfying a modified wave equation. Thus, following Blevins [1], the region $|x| < a$, $-\infty < y < \infty$ is divided into adjacent cells, each cell of volume V containing a single cylinder of volume v so that the solidity factor $\sigma = \frac{v}{V} < 1$. (It is convenient to regard each cell and cylinder as having unit length in the z-direction.) We now replace each rigid fixed cylinder by a cylinder of compressible fluid of density ρ similar to that in the rest of the cell. This 'virtual' cylinder will compress and experience forces on its boundaries. By choosing an appropriate mass flux Q in (5), and an appropriate external body force $\boldsymbol{F}$ in (6), we seek to (i) prevent compression of the cylinder of fluid, and (ii) ensure it remains stationary, thereby modelling the fixed rigid cylinder. Such mass fluxes and body forces will then be smeared out over the whole cell. Now in time δt the volume v of a cylinder of fluid changes by an amount δv where $\delta v/v = -\delta\rho/\rho$ so the rate of change in mass per unit cylinder volume is $(\rho\delta v/v)/\delta t = -\partial\rho/\partial t$ whence the rate of change in mass per unit *cell* volume is $-vV^{-1}\partial\rho/\partial t = -\sigma\partial\rho/\partial t$. Thus we choose a mass flux

$$Q = \sigma \frac{\partial \rho}{\partial t} \tag{11}$$

in (6) in order to counter this compression.

Now the unsteady inertia force on a fluid cylinder per unit volume may be written

$$\boldsymbol{f} = \rho_0(\boldsymbol{I} + \boldsymbol{A})\dot{\boldsymbol{u}}, \ \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{12}$$

where $\boldsymbol{A}$ is the added inertia matrix and I the 2×2 identity matrix. Equation (12) assumes $ka \ll 1$ and wide-spacing between adjacent cylinders. Also for lines of symmetry parallel to x and y, $a_{12} = a_{21} = 0$. So the total force which needs to be applied *by* the fluid cylinder *to* the rest of the fluid in the cell per unit volume of that fluid is

$$\boldsymbol{F} = -\frac{v\boldsymbol{f}}{(V - v)} = -\frac{\sigma}{1-\sigma}\rho_0(\boldsymbol{I} + \boldsymbol{A})\dot{\boldsymbol{u}}. \tag{13}$$

Using (11) and (13) in (5) and (6) gives, using (10),

$$\boldsymbol{\nabla}.\{(\boldsymbol{I} + \sigma\boldsymbol{A})^{-1}\boldsymbol{\nabla}\phi\} + k^2\phi = 0, \quad k = w/c_0$$

or

$$(1 + \sigma a_{22})\phi_{xx} + (1 + \sigma a_{11})\phi_{yy} - \sigma(a_{12} + a_{21})\phi_{xy} + k^2\Delta\phi = 0 \tag{14}$$

where $\Delta = \det(\boldsymbol{I} + \sigma\boldsymbol{A})$.

If we assume lines of symmetry of the columns parallel to x and y, then $a_{12} = a_{21} = 0$ and (14) reduces to

$$(1 + \sigma a_x)^{-1}\phi_{xx} + (1 + \sigma a_y)^{-1}\phi_{yy} + k^2\phi = 0 \quad \text{in } |x| < a \text{ for all } y \tag{15}$$

where $a_{11} \equiv a_x,\ a_{22} \equiv a_y$. Numerical computation reported by Burton [2] suggest that a_x/a_y varies linearly with aspect ratio $\alpha = s_x/s_y$ where s_x, s_y are the spacings between adjacent cylinders in the x and y direction respectively and that $a_x a_y$ varies roughly linearly with σ. However experimental results of Blevins [1] suggest it is sufficient to assume that $a_x = a_y = 1$ so that in $|x| < a$ we have

$$(\nabla^2 + k_1^2)\phi = 0 \tag{16}$$

where $k_1 = k(1+\sigma)^{\frac{1}{2}}$ an increased wavenumber or $c_1 = w/k_1 = c_0(1+\sigma a)^{-\frac{1}{2}}$ a reduced sound speed.

We shall assume (15) is valid in $|x| < a$ and (2) in $|x| > a$, all y and apply continuity of ϕ, ϕ_x at $x = \pm a$.

The Scattering Problem

Symmetry suggests splitting the problem by writing $\phi(x,y) = \phi_s(x,y) + \phi_a(x,y)$ where $\phi_s(\phi_a)$ is even (odd) in x.

Appropriate forms are then

$$\phi_{s,a} = (e^{-i\mathcal{K}x} + R_{s,a}e^{i\mathcal{K}x})e^{i\alpha y} \qquad |x| > a$$

for an incident wave at an angle θ_0 with the negative x direction if $\alpha = k\sin\theta_0$, $\mathcal{K} = k\cos\theta_0$, and

$$\left.\begin{array}{rcl} \phi_s & = & A_s\cos\mathcal{K}_1 x e^{i\alpha y} \quad |x| < a \\ \phi_a & = & A_a \sin\mathcal{K}_1 x e^{i\alpha y} \end{array}\right\} \tag{17}$$

$$\text{if} \quad \mathcal{K}_1 = (1+\sigma a_x)^{\frac{1}{2}}\left\{k^2 - \frac{\alpha^2}{(1+\sigma a_y)}\right\}^{\frac{1}{2}} > 0.$$

The continuity conditions then show that

$$R_s e^{2i\mathcal{K}a} = \frac{i\mathcal{K} - \mathcal{K}_1\tan\mathcal{K}_1 a}{i\mathcal{K} + \mathcal{K}_1\tan\mathcal{K}_1 a}, \quad R_a e^{2i\mathcal{K}a} = \frac{i\mathcal{K} + \mathcal{K}_1\cot\mathcal{K}_1 a}{i\mathcal{K} - \mathcal{K}_1\cot\mathcal{K}_1 a} \tag{18}$$

Trapped modes

Trapped modes can exist in $|x| < a$ if $k < \alpha < (1+\sigma ay)^{\frac{1}{2}}k$ since now $\mathcal{K} \equiv i\beta$ $= i(\alpha^2 - k^2)^{\frac{1}{2}}$, $\beta > 0$ and, consistent with (17) in $|x| < a$ are solutions

$$\phi_{s,a} = e^{-\beta x}e^{i\alpha y} \quad \text{in} \quad |x| > a \tag{19}$$

whence continuity requires

$$\beta = \mathcal{K}_1\tan\left(\mathcal{K}_1 a - \frac{j\pi}{2}\right) \quad \begin{array}{rcl} j & = & 0, \text{ even solution} \\ & = & 1 \text{ odd solution} \end{array} \tag{20}$$

and

$$\frac{\mathcal{K}_1^2}{(1+\sigma a_x)} + \frac{\beta^2}{(1+\sigma a_y)} = \sigma a_y k^2. \tag{21}$$

A sketch of (20), (21) in the $\beta, \mathcal{K}_1$ plane makes clear that for each value of

$$\lambda \equiv \sigma a_y(1+\sigma a_x)^{\frac{1}{2}}k \tag{22}$$

there exist pairs $(\beta_i, \mathcal{K}_{1i})$ $i = 1, 2, \ldots, n$ corresponding to trapped modes and

$$(r-1)\frac{\pi}{2a} < \mathcal{K}_{1r} < \frac{r\pi}{2a}, \quad r = 1, 2, \ldots, n$$

whenever

$$(n-1)\frac{\pi}{2a} < \lambda < \frac{n\pi}{2a}.$$

Returning to the water-wave problem, given a frequency $w/2\pi$ we can determine k from $w^2 = gk \tanh kh$ and hence λ from (22) which in turn determines β_i from which the trapped mode wavenumbers α_i along the y-direction can be derived from

$$\alpha_i = (\beta_i^2 + k^2)^{\frac{1}{2}}.$$

Excitation of trapped modes

It is not possible to excite these trapped modes by a wave incident from infinity since in that case $\alpha = k \sin \theta_0 < k$. However a localised source of energy *can* be shown to excite trapped modes. For example the presence of a sharp headland could act as a radiator of an incident wave towards $|x| < a$. Thus a simple model is to assume that in $|x| > a$ the governing equation is $\phi_{xx} + \phi_{yy} + k^2\phi = -\delta(x-b)\delta(y)$ i.e. there is a line source at $x = b$, $y = 0$. We can now ask how much energy from this source is fed into trapped modes producing possibly large modes of oscillation, and how much gets radiated out to sea? A related problem in electromagnetic theory is discussed in Collin [3].

The mathematical solution to this problem is simply

$$\phi(x,y) = -\frac{1}{4\pi i} \int_C \left(\frac{2e^{i\mathcal{K}|x-b|}}{\mathcal{K}} + \frac{(R_s + R_a)}{\mathcal{K}} e^{i\mathcal{K}(x+b)} \right) e^{-i\alpha y} d\alpha \tag{23}$$

where C is a contour chosen to satisfy appropriate radiation conditions. The contribution to the trapped modes arises from the poles of the integrand or the zeros of the denominators of R_s, R_a in (18) whilst the contribution to the energy radiated away from the columns out to sea, arises from the branch cuts at $\alpha = \pm k$.

Further details of the derivation of the solution (23) and its implications will be presented at the workshop.

References

[1] Blevins, R.D. Acoustic modes of heat exchanges tube bundles. *J. Sound and Vibration*, 1986, **109** (1), 19-31.

[2] Burton, T.E. Sound speed in a heat exchanger tube bank. *J. Sound and Vibration*, 1980, **71** (1), 157-160.

[3] Collin, R.E. *Field Theory of Guided Waves*, IEEE Press, 1991 p. 725.