SOLITARY WAVES IN A STRATIFIED FLUID: MODELLING AND EXPERIMENTS

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This study concerns solitary waves in a stratified fluid and is motivated by possible effects of sub-surface waves in a layered ocean on compliant offshore units. Such offshore units may be floating platforms or ships at the sea surface with connecting risers and cables to wells and equipment at the sea floor. The concern is the possible loads and induced vibrations of the risers and cables caused by sub-surface waves.

The ocean often has a shallow upper layer with a stratification that is relatively close to a linear function and a deep lower layer with constant density. This motivates for developing a two-layer model where the Brunt-Väisälä frequency is constant in the upper layer and zero in the lower. The basic equations of the fully nonlinear model are derived along the lines of previous studies, most notably Yih (1960) and Turkington et al. (1991). Coordinates $O-xy$ are introduced, with the x-axis horizontal and the y-axis vertical, and with unit vectors $\mathbf{i}$ and $\mathbf{j}$ accordingly. We consider motion in two dimensions where waves of permanent form are propagating with speed c horizontally in the fluid. Viewing the problem in a frame of reference which follows the waves, the motion becomes steady, with a horizontal current with speed c along the negative x-axis in the far-field. The undisturbed fluid has a vertical density profile

$$\rho(y) \;=\; \begin{cases} \rho_0 - \Delta\rho\, y\,/h_2, & \text{for} \qquad 0 < y < h_2, \\ \rho_0, & \text{for} \quad -h_1 < y < 0, \end{cases} \tag{1}$$

We assume that the fluid is incompressible and inviscid. The former means that $\nabla\cdot\mathbf{v}=0$ where $\mathbf{v}=(u,v)$ denotes the fluid velocity. Conservation of mass, $\nabla\cdot(\rho\mathbf{v})=0$, then gives that $\mathbf{v}\cdot\nabla\rho=0$. Following the procedure of Yih (1960) we introduce pseudo velocities $u'=(\rho/\rho_0)^{1/2}u$, $v'=(\rho/\rho_0)^{1/2}v$. Furthermore we introduce a pseudo stream function Ψ' such that $\mathbf{v}'=\nabla\Psi'\times\mathbf{k}$ where $\mathbf{k}=\mathbf{i}\times\mathbf{j}$. It follows that $\rho=\rho(\Psi')$. From the equations of motion the following relation may be derived

$$\rho_0\nabla^2\Psi' + gy\frac{d\rho}{d\Psi'} = \frac{dH(\Psi')}{d\Psi'} = h(\Psi'), \tag{2}$$

where $-\nabla^2\Psi'$ determines the pseudo vorticity and $H=p+\frac{1}{2}\rho(u^2+v^2)+\rho gy$ is the Bernoulli constant being conserved along a streamline determined by $\Psi'=$ constant. Furthermore, p denotes pressure and g the acceleration due to gravity. $dH/d\Psi'$ is determined in the far-field, giving

$$\frac{dH}{d\Psi'} = \Big(\frac{dp}{dy}+\rho g\Big)\frac{dy}{d\Psi'} + \frac{c^2}{2}\frac{d\rho}{d\Psi'} + gy\frac{d\rho}{d\Psi'}, \tag{3}$$

The vertical component of the equation of motion becomes in the far-field $p_y+\rho g=0$, which means that the first term on the right of (3) is zero. The pseudo stream function is then decomposed by $\Psi'=\Psi'_\infty+\psi'$, where Ψ'_∞ satisfies

$$\frac{d\Psi'_\infty}{dy} = -c\left(\frac{\rho}{\rho_0}\right)^{1/2} \tag{4}$$

giving

$$\nabla^2\Psi'_\infty = \frac{c^2}{2\rho_0}\frac{d\rho}{d\Psi'}. \tag{5}$$

Since $d\rho/d\Psi'$ is constant along each streamline, (2) becomes

$$\rho_0\nabla^2\psi' + g(y-y_\infty)\frac{d\rho}{d\Psi'} = 0, \tag{6}$$

where y and y_∞ are vertical coordinates on the same streamline, with y_∞ in the far-field.

¿From now on we apply the Boussinesq approximation, i.e. exploit that $\Delta\rho/\rho$ is small. Integrating (4) we find $\Psi'_\infty = -cy[1+O(\Delta\rho/\rho)]$, giving $y - y_\infty = \psi'/c$. Furthermore we have

$$\frac{g}{\rho_0}\frac{d\rho}{d\Psi'} = \frac{g}{\rho_0}\frac{d\rho}{dy}\frac{dy}{d\Psi'} \simeq \frac{N^2}{c}[1+O(\Delta\rho/\rho)], \tag{7}$$

where $N^2 = -(g/\rho_0)(d\rho/dy)$ determines the Brunt-Väisälä frequency. Within the Boussinesq approximation we may also replace the pseudo stream function by the stream function Ψ such that $\mathbf{v} = \nabla\Psi\times\mathbf{k}$. Correspondingly, Ψ'_∞ and ψ' are replaced by Ψ_∞ and ψ, respectively. Let $\psi = \psi_2$ in the upper layer and $\psi = \psi_1$ in the lower. Then ψ_2 satisfies the Helmholtz equation in the upper layer, i.e.

$$\nabla^2\psi_2 + \frac{N^2}{c^2}\psi_2 = 0\,. \tag{8}$$

ψ_1 satisfies the Laplace equation in the lower layer, i.e.

$$\nabla^2\psi_1 = 0\,. \tag{9}$$

The upper boundary of the upper layer is a free surface. With $\Delta\rho/\rho$ small this boundary may be approximated by a horizontal rigid lid. We assume that the bottom of the lower layer is horizontal at $y = -h_1$. Thus, $\psi_2 = 0$ at $y = h_2$, $\psi_1 = 0$ at $y = -h_1$. The two layers are separated by the streamline with vertical coordinate η where $\eta \to 0$ for $x \to \pm\infty$. The kinematic boundary condition requires that the fluid velocity is continuous at the boundary between the layers, i.e. that

$$\nabla(\psi_1 - cy) = \nabla(\psi_2 - cy) \quad \text{at} \quad y = \eta. \tag{10}$$

The formulation is fully nonlinear, where the stream functions $\psi_{1,2}$, the stream line η and the wave speed c shall be determined.

We solve the nonlinear problem (8)–(10) by means of integral equations and introduce two Green functions G_1 and G_2. The first Green function is a pole at $(x,y) = (x',y')$ and satisfies the Laplace equation (9), i.e.

$$G_1(x,y,x',y') = \ln\frac{r}{r_1}\,. \tag{11}$$

The second Green function is a pole at $(x,y) = (x',y')$ and satisfies the Helmholtz equation (8), i.e.

$$G_2(x,y,x',y') = \frac{\pi}{2}[Y_0(Kr) - Y_0(Kr_2)]\,, \tag{12}$$

where Y_0 denotes the Bessel function of second kind of order zero and $K = N/c$. Furthermore,

$$r = [(x-x')^2 + (y-y')^2]^{1/2}, \quad r_{1,2} = [(x-x')^2 + (y+y' \pm 2h_{1,2})^2]^{1/2}. \tag{13}$$

The stream functions are determined by

$$\psi_{1,2} = \int_I \sigma_{1,2}(s')G_{1,2}(x,y,x'(s'),y'(s'))ds'\,, \tag{14}$$

where $\sigma_1(s)$ and $\sigma_2(s)$ denote yet unknown distributions, I denotes the contour $y = \eta$ and s arclength.

Choosing h_2 as length scale and the linear long wave speed c_0 as velocity scale (and h_2/c_0 as time scale) we determine Nh_2/c_0 from a linearized set of equations. Then the nondimensional quantities Kh_2, σ_1/c_0, σ_2/c_0, η/h_2 and c/c_0 depend on the parameters h_1/h_2 and η_{max}/h_2, and not on $\Delta\rho/\rho$. Thus, within the Boussinesq approximation, the actual value of $\Delta\rho/\rho$ enters only in the problem through c_0. Velocity profiles are visualized in figure 1.

The investigation combines theory and experiments. The latter are performed in a wave tank with a stable two-layer fluid with a shallow fluid of linear stratification above or below a deeper fluid of homogeneous density. We generate solitary waves of mode one which propagate along the wave tank. The amplitude of the waves, defined by the maximal excursion of the stratified layer, is in a rather large range. Particle tracking velocimetry (PTV) and particle image velocimetry (PIV) are employed to make detailed recordings of the induced velocities due to the waves.

Particular focus is paid to the role of the breaking of the waves observed in the experiments. For the large waves the breaking occurs in a region in the centre of the waves in the thin layer with linearly stratified fluid. The breaking serves to limit the fluid velocity. The latter is in the region with breaking found to be of the form $\mathbf{v} = c\mathbf{i} + \mathbf{v}'$ where $c\mathbf{i}$ denotes the wave velocity and $\mathbf{v}'$ a velocity field where $|\mathbf{v}'| << c$. This means that the wave speed provides an upper bound of the fluid velocity induced by the wave, practically speaking. The wave breaking occurs similarly in the experiments with the inverted two-layer model, when the waves are large. A fluid velocity approximately equal to the wave speed means that the wave transports mass.

The experimental and theoretical velocity fields exhibit good agreement up to breaking, generally speaking. Intensive breaking is found to occur for a wave amplitude a less than about 0.8 times the depth h_2 of the linearly stratified layer. This is in agreement with the theoretical model which predicts an induced fluid velocity being less than the wave speed when $a/h_2 < 0.855$. For larger waves the theory does not fit with the experimental observations. We find that the experimental waves broaden when the non-dimensional wave amplitude exceeds 0.8–0.9 (figure 2). The experiments suggest that the broadening is caused by the wave breaking which limits the magnitude of the fluid velocity. The broadening is not reproduced by the theory. The broadening effect found here is entirely different from the one taking place in a two-fluid system with constant densities in each of the layers.

In the present two-layer model a solitary wave always exhibits an excursion out of the layer with linear stratification.

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References

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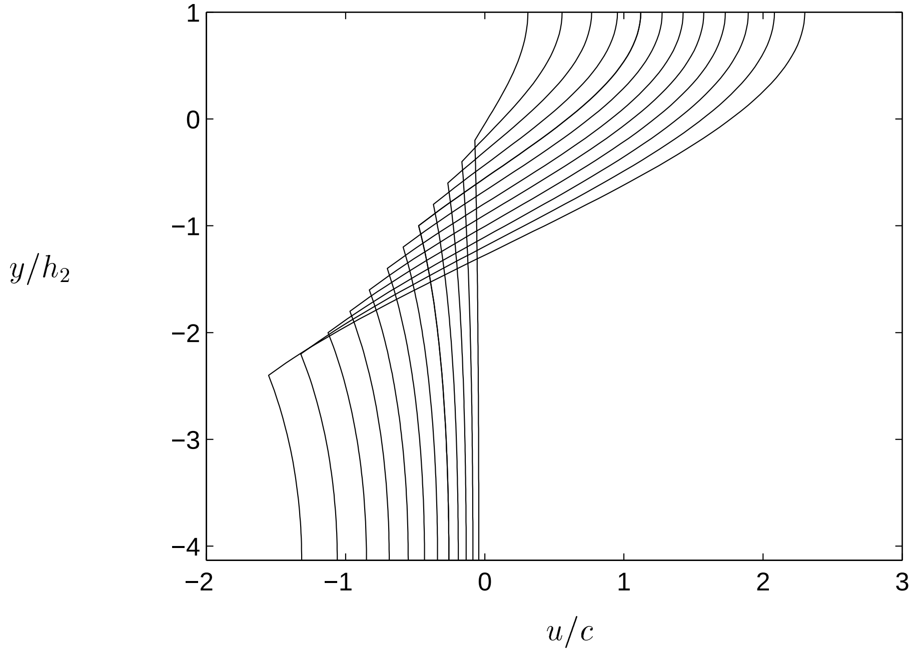

Figure 1: Velocity profiles at wave crest. $a/h_2 = 0.2, 0.4, 0.6, ..., 2.4$. $h_1/h_2 = 4.13$.

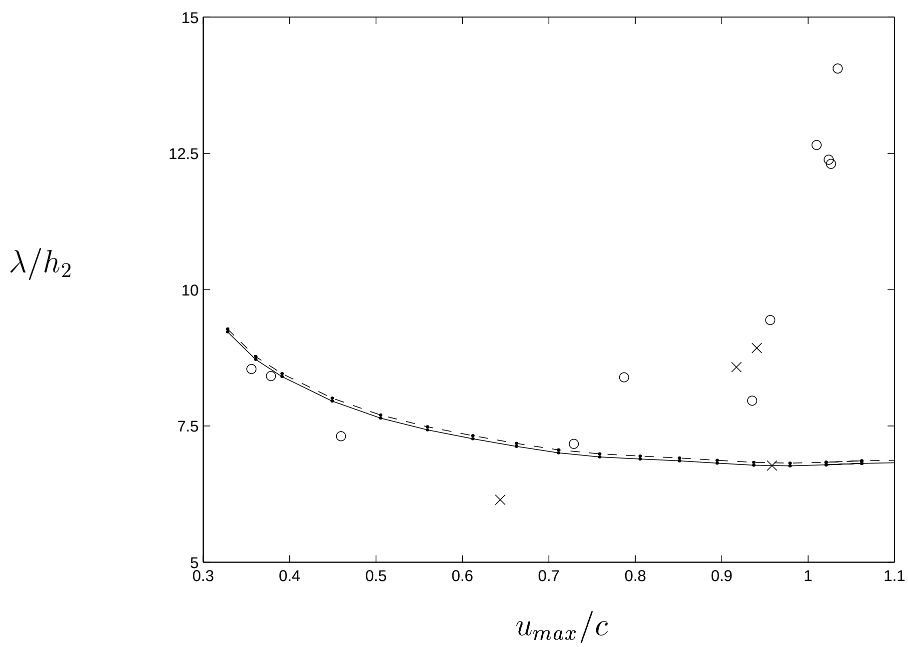

Figure 2: Nondimensional wave length λ/h_2 vs. nondimensional velocity u_{max}/c. Small circles and crossed (experiments) Solid and dashed line (theory).