EMBEDDED TRAPPED MODES FOR ARRAYS OF CYLINDERS

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Introduction

In the problem of the scattering of water waves by a regularly-spaced, finite line of cylinders, it has been observed (see, for example, Maniar & Newman, 1997) that large hydrodynamic forces on individual cylinders occur at frequencies which are close to the trapped-mode frequencies associated with an infinitely-long line of the same spacing. For cylinders of uniform cross-section, these frequencies are determined by reducing the problem to a two-dimensional channel problem by invoking the periodicity of the trapped mode solution and removing the channel depth and time dependence from the problem. Trapped modes are thus non-trivial solutions to the Helmholtz equation in the channel region exterior to the body that satisfy homogeneous boundary conditions and have finite energy. For a fairly general class of cylinders symmetrically placed on the centre-line of the channel, Evans, Levitin & Vassiliev (1994) proved that trapped modes exist at frequencies below the first cut-off for antisymmetric wave propagation down the channel. The solutions are antisymmetric about the centre line of the channel.

More recently Evans & Porter (1998) have provided numerical evidence for the existence of an antisymmetric mode trapped by a circular cylinder on the centre line of the channel, for which the frequency is *above* the lowest cut-off frequency for antisymmetric motion. In other words this mode is embedded in the continuous spectrum. This mode occurs only for a cylinder of one specific radius; this is in contrast to the modes at frequencies below the cut-off which occur for cylinders of any radius. The purpose of this work is to show that this mode is not isolated but is one of a continuous branch of modes which exist for ellipses of continuously varying aspect ratio. A similar branches has also been found for modes supported by rectangular blocks and it is anticipated that such branches exist for a wide class of bodies which are described by two geometric parameters. For the ellipses, the branch of modes begins with a flat plate of specific length aligned along the channel, and ends with a flat plate of a different specific length which is perpendicular to the channel walls. Approximations may be found for the length of the plate aligned with the walls and the trapped mode frequency associated with it. The trapped-mode frequency tends to the second cut-off for antisymmetric motion when the body approaches a perpendicular plate, and an exact asymptotic result for the length of this plate is presented.

Formulation

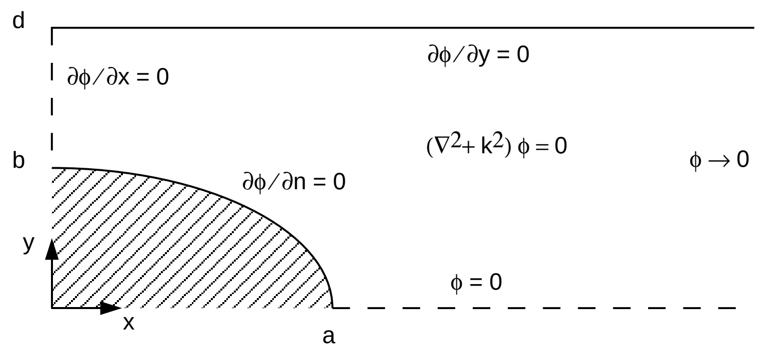

Figure 1: Definition sketch.

A quarter section of the channel and body is illustrated in figure 1 together with the details of the boundary-value problem. The wavenumber k, which is to be found, is directly related to the fluid oscillation frequency. Trapped modes are sought which are symmetric in x, antisymmetric in y, and which have frequencies between the first and second cut-off for antisymmetric wave propagation so that $\pi/2 < kd < 3\pi/2$. Thus the trapped mode potential ϕ satisfies the Helmholtz equation and has zero normal derivative on the quarter body, the channel wall $y = d$ and the line of symmetry $x = 0$. In addition $\phi = 0$ on the centre line of the channel $y = 0$ and $\phi \to 0$ as $x \to \infty$.

The method of solution is as follows. A complex, homogeneous integral equation is derived for the trapped mode potential ϕ. This is split into real and imaginary parts and the resulting equations are discretised numerically. A non-zero solution for ϕ which decays at infinity is possible if the determinant of the real matrix is zero and if the eigenvector of this matrix which corresponds to the zero eigenvalue satisfies the imaginary part of the equation. The latter is regarded as a side condition which ϕ must satisfy.

The Green's function for the channel which satisfies the same boundary conditions as ϕ on $y = 0$, $y = d$ and $x = 0$ can be written

$$G = \frac{1}{4}[Y_0(kr) + Y_0(kr_1) + Y_0(kr_2) + Y_0(kr_3)] + \frac{2}{\pi}\,\mathrm{Re} \,⨍_0^\infty \frac{e^{-k\gamma d}\cosh k\gamma(d-y)}{\gamma\cosh k\gamma d} \cos kxt \cos k\xi t \,\mathrm{d}t$$

$$- \frac{2i}{kd}\sum_{n=1}^{j_a} \frac{1}{\tau}\sin\left(n-\frac{1}{2}\right)\frac{\pi y}{d}\sin\left(n-\frac{1}{2}\right)\frac{\pi\eta}{d}\cos kx\tau_n \cos k\xi\tau_n, \tag{1}$$

where

$$r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}, \quad r_1 = [(x-\xi)^2 + (y+\eta-2d)^2]^{1/2}, \tag{2}$$

$$r_2 = [(x+\xi)^2 + (y-\eta)^2]^{1/2}, \quad r_3 = [(x+\xi)^2 + (y+\eta-2d)^2]^{1/2}, \tag{3}$$

$$\gamma(t) = \begin{cases} -\mathrm{i}(1-t^2)^{1/2}, & t \le 1, \\ (t^2-1)^{1/2}, & t > 1, \end{cases} \tag{4}$$

$$\tau_n = [1 - ((n-1/2)\pi/kd)^2]^{1/2} \tag{5}$$

and

$$(j_a - 1/2)\pi < kd < (j_a + 1/2)\pi. \tag{6}$$

If the frequency is below the first cut-off for antisymmetric modes, that is $kd < \pi/2$, then from (6) $j_a = 0$ and from (1) the Green's function is purely real and decays as $|x| \to \infty$. This allows a real integral equation to be developed to search for trapped modes in this region. However, if $\pi/2 < kd < 3\pi/2$ then $j_a = 1$ and the Green's function is complex and radiates waves to infinity. In particular

$$\mathrm{Im}[G] = -\frac{2}{kd\tau_1}\sin\frac{\pi y}{2d}\sin\frac{\pi\eta}{2d}\cos k\xi\tau_1\cos kx\tau_1. \tag{7}$$

Application of Green's theorem to ϕ and G yields

$$\frac{1}{2}\phi(x,y) = \int_{\partial D} \phi(\xi,\eta)\frac{\partial G}{\partial n}(x,y;\xi,\eta)\,\mathrm{d}S, \tag{8}$$

where ∂D is the boundary of the body in the region $x \ge 0$, $y \ge 0$, (x, y) is a point on the cylinder surface, and $\partial G/\partial n$ is the inward normal derivative of G to the body with respect to the variables (ξ, η). Without loss of generality the function ϕ is assumed to be real and so (8) reduces to the two equations

$$\frac{1}{2}\phi(x,y) = \int_{\partial D} \phi(\xi,\eta)\frac{\partial}{\partial n}[\mathrm{Re}[G(x,y;\xi,\eta))]]\,\mathrm{d}S, \tag{9}$$

and

$$\int_{\partial D} \phi(\xi, \eta) \frac{\partial}{\partial n} [\mathrm{Im}[G(x, y; \xi, \eta))]] \, \mathrm{d}S_q = 0. \tag{10}$$

The integral in (9) is discretised and using collocation a matrix system in the form

$$\sum_{j=1}^{n} M_{ij} \phi_j = 0, \tag{11}$$

is obtained, where $\phi_j = \phi(\xi_j, \eta_j)$ and (ξ_j, η_j), $j = 1, \ldots, n$, are the collocation points. From (7) the imaginary part of the equation may be discretised to give the side condition

$$S = \sum_{j=1}^{n} \phi_j \frac{\partial}{\partial n} \left[\sin \frac{\pi \eta}{2d} \cos k\xi\tau_1 \right]_{(\eta,\xi)=(\eta_j,\xi_j)} \mathrm{d}S_j = 0. \tag{12}$$

The system of equations in (11) has a non-zero solution for ϕ when $\det(M) = 0$. However this solution only corresponds to a trapped mode if the resulting function ϕ satisfies the side condition (12). Suppose the body is an ellipse with semi-major axis a and semi-minor axis b. For a fixed value of b/a the curve on which $\det(M) = 0$ is plotted in the a/d-kd plane. Rather than look for points on this line on which $S = 0$, the method of Evans & Porter (1998) is followed. At points at which $\det(M) \neq 0$, M has at least one eigenvalue of smallest magnitude, λ say, with corresponding normalised eigenvector ψ. If λ is real then ψ may be taken to be real whereas if λ is complex then ψ must be complex. Nonetheless the curve

$$\tilde{S} = \mathrm{Re} \sum_{j=1}^{n} \psi_j \frac{\partial}{\partial n} \left[\sin \frac{\pi \eta_j}{2d} \cos k\xi_j\tau_1 \right]_{(\eta,\xi)=(\eta_j,\xi_j)} \mathrm{d}S_j = 0 \tag{13}$$

may be drawn in the a/d-kd plane. At the point where the curve $\tilde{S} = 0$ crosses the curve $\det(M) = 0$, $\lambda = 0$ and so $\psi = \phi$ and $S = \tilde{S} = 0$ and the point corresponds to a trapped mode. Numerically this is a more robust method for the calculation of the trapped mode frequencies and points than looking for zeros of the complex determinant because it is found that the curves on which the real and imaginary part of the determinant are zero touch rather than cross in the a/d-kd plane.

Results and discussion

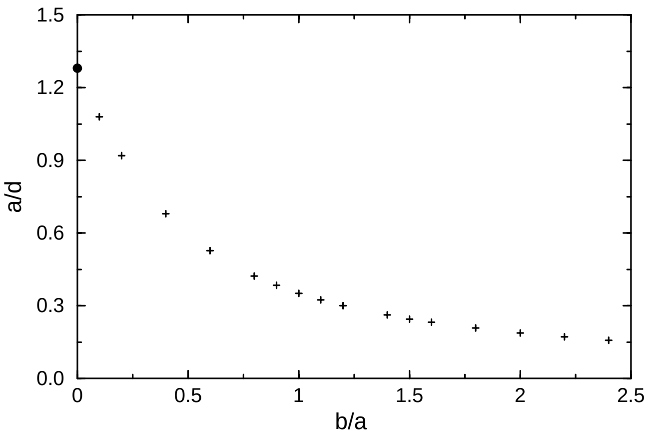

Figure 2: Ellipse geometries that are able to support trapped modes.

Results for embedded trapped modes with $kd \in (\pi/2, 3\pi/2)$ found using the method described in the previous section are shown in figures 2 and 3. These calculations are for ellipses with axis lengths $2a$ in the x direction and $2b$ in the y direction. It is already known (see Evans & Porter, 1998) that an embedded trapped mode exists for the special case of a circle with radius

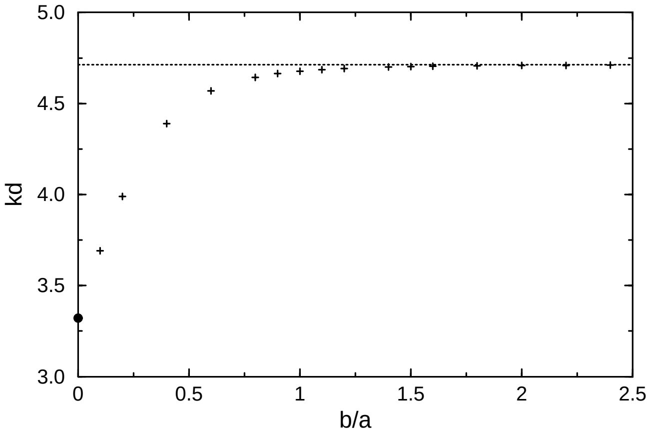

Figure 3: Wave numbers of trapped modes supported by ellipses.

$a/d = b/d \approx 0.352$; the corresponding frequency is given by $kd \approx 1.489\pi$. Figure 2 indicates that the trapped mode for this circle is one of a branch of trapped modes for ellipses with $b/a \in [0, \infty)$. The corresponding trapped-mode wave numbers kd are shown in figure 3. The cut-off at $kd = 3\pi/2$ is indicated by the dashed line and it can be seen that this cut-off is approached as $b/a \to \infty$.

The limiting case $b/a \to 0$ corresponds to a thin plate of a specific length aligned with the channel walls. The trapped mode for this problem has been computed by an approximate procedure based on the assumption that the plate is long and it is found that $a/d \approx 1.28$ and $kd \approx 1.06\pi$; these points are marked on the figures by filled circles. The limiting case $b/a \to \infty$ corresponds to a thin plate perpendicular to the channel walls. An asymptotic analysis may be performed for this case and it is found that the limiting value of b/d is a root of

$$J_1(2\pi b/d) = J_1(\pi b/d). \tag{14}$$

There is only one root for $b/d \in (0, 1)$ with the value $b/d \equiv B \approx 0.392$. The asymptotic analysis also yields the result

$$kd = \frac{3\pi}{2} - \frac{3\pi^3 a^2}{16d^2} \left[J_1\left(3\pi B\right)\right]^2 + O(a^4/d^4) \quad \text{as} \quad a/d \to 0 \tag{15}$$

giving the approach to the cut-off illustrated in figure 3.

Further calculations, not shown here, indicate the existence of further branches of embedded trapped modes for elliptical structures that emanate from additional trapped-mode solutions for a flat plate aligned with the channel walls. All of these branches terminate at $kd = 3\pi/2$ for some finite $b/a < 1$. Calculations for a rectangular block are also in progress.

Acknowledgements

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