Steady Free-Surface Flow in Water of Finite Depth

Xiao-Bo Chen[1] & Rong Zhao[2]

[1]Bureau Veritas, DTA, 17bis, Place des Reflets, 92400 Courbevoie (France)
Fax: 33-1-4291.3395 Email: xiao-bo.chen@bureauveritas.com

[2]Marintek, Otto Nielsensv.10, PO Box 4125, N-7450 Trondheim (Norway)
Fax: 47-7359.5870 Email: rong.zhao@marintek.sintef.no

The steady free-surface potential flow generated by a source advancing at constant horizontal speed is considered by making use of the formulations developed by Chen & Nguyen (2000) in water of finite depth. Further to that work on the singular and highly-oscillatory properties of the steady free-surface potential near the track of the source located close to or at the free surface, another peculiar property is analyzed here. It is shown that the classical formulation suffers a non-uniformity when the waterdepth tends to infinity. A uniform formulation is then found by extracting the constant terms - one of them being infinity! Furthermore, the wave component of steady flow in subcritical and supercritical regimes is presented.

1. Green function of steady flow in water of finite depth

Under the reference system moving with the source at the speed U along the positive x-axis defined by its (x, y) plane coinciding with the mean free surface and z-axis oriented positively upward, the ship-motion Green functions $G(\vec{\xi}, \vec{x}_s)$ representing the velocity potential of the flow created at a point $\vec{\xi} = (\xi, \eta, \zeta)$ by a steadily-advancing source of unit strength located at a point $\vec{x}_s = (x_s, y_s, z_s)$, can be expressed as

$$G = G^S + G^F \tag{1}$$

where G^F accounts for free-surface effects and G^S is defined in terms of simple singularities

$$4\pi\, G^S = \sum_{n=-\infty}^{\infty} (-1)^n \left\{ -1/\sqrt{r^2 + (\zeta - z_s + 2nh)^2} + 1/\sqrt{r^2 + (\zeta + z_s + 2nh)^2} \right\} \tag{2}$$

in which $r = \sqrt{(\xi - x_s)^2 + (\eta - y_s)^2}$ and $h = H/L$ is the adimensional waterdepth with respect to the reference length L. The simple part G^S defined by (2) satisfies $G^S = 0$ at the free surface ($\zeta = 0$) and $\partial G^S/\partial\zeta = 0$ at the sea bed ($\zeta = -h$). The free-surface part G^F in (1) is defined by a double integral representing the Fourier superposition of elementary waves

$$4\pi^2 G^F = \lim_{\epsilon \to +0} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \, \frac{A\, e^{-i(\alpha x + \beta y)}}{D + i\epsilon \Sigma_1} \tag{3}$$

with $(x, y) = (\xi - x_s, \eta - y_s)$ and A defined by

$$A = \cosh k(\zeta + h) \cosh k(z_s + h) / \cosh^2 kh \quad \text{with} \quad k = \sqrt{\alpha^2 + \beta^2} \tag{4}$$

Furthermore, the dispersion function D in (3) is given by

$$D = F^2 \alpha - k \tanh kh \tag{5}$$

in which $F = U/\sqrt{gL}$ is Froude number with g the acceleration of gravity. The function Σ_1 in (3) is given by $\Sigma_1 = -\operatorname{sign}(\alpha)$ and is significant in the region $D(\alpha, \beta) \approx 0$.

If we use the polar Fourier variables (k, θ), the equation (3) becomes

$$4\pi^2 G^F = \lim_{\epsilon \to +0} \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} dk \, \frac{A\, e^{-ik(x\cos\theta + y\sin\theta)}}{D/k - i\epsilon \operatorname{sign}(\cos\theta)} \tag{6}$$

in which A is given by (4), and by using (5) D/k becomes :

$$D/k = k[F^2 \cos^2\theta - \tanh(kh)/k] \tag{7}$$

Due to the symmetry properties of the dispersion function $D(k, \theta)$, it can be easily verified that the free-surface part of the Green function G^F given by (6) is symmetrical with respect to the axis $y = 0$ and that its imaginary part is nil. The connection to the conventional form (eq.13.37 in Wehausen & Laitone, 1960) can be obtained by using the residue theorem and exploiting the symmetrical properties of the Fourier integral.

2. Non-uniformity of the Green function G^F

The free-surface component G^F is defined by the double Fourier integral (6) in which the amplitude function A and the dispersion function D are dependent on the waterdepth h as a parameter. It can be easily verified that the limit of the amplitude function $A(k,h)$ given by (4)

$$\lim_{k\to 0}\left\{\lim_{h\to\infty} A(k,h)\right\} = \lim_{h\to\infty}\left\{\lim_{k\to 0} A(k,h)\right\} = 1 \tag{8}$$

However, as far as the dispersion function $D(k,h)$ is concerned, we have from (7)

$$\lim_{k\to 0}\left\{\lim_{h\to\infty} D(k,h)\right\} = k(F^2 k\cos^2\theta - 1) = O(k) \tag{9a}$$

and

$$\lim_{h\to\infty}\left\{\lim_{k\to 0} D(k,h)\right\} = k^2(F^2\cos^2\theta - h) = O(-k^2 h) \tag{9b}$$

by using the development $\tanh(t) = t + O(t^3)$ as $t = kh \to 0$ in (7).

If the limit (9a) is used, we usually say that G^F given by (6) tends to the Green function in deep water. However, as the Fourier variable k is involved in the integral representation of G^F (6) and if the limit (9b) should be used, different values of G^F may be obtained since

$$\lim_{h\to\infty}\left\{\lim_{k=h^{-p}\to 0} D(k,h)\right\} = \begin{cases} 0 & p>1/2 \\ -1 & p=1/2 \\ -\infty & p<1/2 \end{cases} \tag{10}$$

This non-uniform behavior of the dispersion function implies that the classical expression of the Green function G^F given by (6) may not be uniform in the limit of deep water, and that some undesirable terms such as constants may be embedded in (6). To identify them, we perform a local analysis of the Green function G^F corresponding to the Fourier integral in the region $k < \delta \ll 1$. We write

$$4\pi^2 G^F_\delta = \int_{-\pi}^{\pi} d\theta \int_0^{\delta} dk\, \frac{A_0}{D_0/k}\, e^{-ik(x\cos\theta + y\sin\theta)} \tag{11}$$

in which

$$A_0 = \left[A(k,h) + (e^{-k\sigma} - 1)\right]|_{k\to 0} = e^{-k\sigma}[1 + O(\delta^2)] \tag{12a}$$

with σ a positive real parameter and

$$D_0/k = D(k,h)/k|_{k\to 0} = k(F^2\cos^2\theta - h) + O(\delta^3) \tag{12b}$$

The parameter σ is introduced to facilitate the analysis as we will extend $\delta \to \infty$, and for the sake of numerical evaluation of the resultant integral. The value of σ is hoped to not affect on the results of the analysis. It will be shown that the undesirable terms are indeed independent of σ.

Introducing above expressions into (11), G^F_δ may be estimated as

$$\begin{aligned} 4\pi^2 G^F_0 = 4\pi^2 G^F_\delta|_{\delta\to\infty} &= \int_{-\pi}^{\pi} \frac{d\theta}{F^2\cos^2\theta - h} \lim_{\kappa\to 0} \int_\kappa^\infty dk\, \frac{e^{-k[\sigma + i(x\cos\theta + y\sin\theta)]}}{k} \\ &= \int_{-\pi}^{\pi} \frac{d\theta}{F^2\cos^2\theta - h} \lim_{\kappa\to 0} E_1\{\kappa[\sigma + i(x\cos\theta + y\sin\theta)]\} \end{aligned} \tag{13}$$

where $E_1\{\cdot\}$ is the exponential integral function defined in Abramowitz & Stegun (1967). At $\kappa \to 0$, the exponential integral function can be written

$$E_1\{\kappa[\sigma + i(x\cos\theta + y\sin\theta)]\} = -\log\kappa - \gamma - \log[\sigma + iR\cos(\theta - \phi)] + O(\kappa) \tag{14}$$

where $\gamma = 0.57721\ldots$ is Euler's constant, and (R,ϕ) defined as

$$R = \sqrt{x^2 + y^2} \quad \text{and} \quad \phi = \arctan(y/x)$$

Using (14) in (13), we have

$$4\pi^2 G^F_0 = C_\infty + C_0 + 4\pi^2 \tilde{G}^F_0 \tag{15}$$

with

$$C_\infty = \lim_{\kappa\to 0} \frac{2\pi \log\kappa}{h\sqrt{1-F^2/h}} \quad \text{for} \quad F/\sqrt{h}<1 \quad ; \qquad C_\infty = 0 \quad \text{for} \quad F/\sqrt{h}>1 \tag{16a}$$

$$C_0 = \frac{2\pi\,\gamma}{h\sqrt{1-F^2/h}} \quad \text{for} \quad F/\sqrt{h}<1 \quad ; \qquad C_0 = 0 \quad \text{for} \quad F/\sqrt{h}>1 \tag{16b}$$

$$4\pi^2 \tilde{G}_0^F = -\int_{-\pi}^{\pi} \frac{\log[\sigma + iR\cos(\theta-\phi)]}{F^2\cos^2\theta - h}\,d\theta \tag{16c}$$

Both C_∞ and C_0 are considered as constant since they are independent of the variables (ξ,η,ζ) and (x_s,y_s,z_s). They are zero in the supercritical regime $(F/\sqrt{h}>1)$ but not in the subcritical regime $(F/\sqrt{h}<1)$. Similar to the dispersion function analyzed in (10), the constant C_∞ at $F/\sqrt{h}<1$ given by (16a) is non-uniform and is equal to infinity if h is finite. The constant C_0 (16b) is finite and disappear in deep water while the term $\tilde{G}_0^F$ is represented by a single integral (16c).

3. Uniform formulation of the Green function

Further to the foregoing analysis, we may now define a uniform formulation of the Green function by simply subtracting the constant terms C_∞ and C_0 from the free-surface component G^F :

$$G = G^S + \bar{G}^F \quad \text{with} \quad \bar{G}^F = G^F - (C_\infty + C_0)/(4\pi^2) = \tilde{G}^F + \tilde{G}_0^F \tag{17}$$

with the new free-surface component $\bar{G}^F$ defined by

$$\tilde{G}^F = G^F - G_0^F = \lim_{\epsilon\to+0} \int_{-\pi}^{\pi} d\theta \int_0^\infty dk\, \mathcal{A}\, e^{-ik(x\cos\theta + y\sin\theta)} \tag{18}$$

with

$$\mathcal{A} = \frac{A}{D/k - i\epsilon\,\mathrm{sign}(\cos\theta)} - \frac{\exp(-k\sigma)}{k(F^2\cos^2\theta - h)} \tag{19}$$

It can be shown that $\mathcal{A} = O(1)$ at $k\to 0$ and has uniform limit for $h\to\infty$.

4. Wave component of the Green function

Following the analysis by Noblesse & Chen (1995), the free-surface part G^F can be decomposed as the sum of a wave component and a nonoscillatory component significant only in the near field. The wave component is defined by the single Fourier integral along the dispersion curves defined by the dispersion relation $D=0$

$$4\pi i\, G^W = \sum_{D=0} \int_{D=0} ds\; (\Sigma_1 + \Sigma_2) e^{-i(\alpha x + \beta y)} A/\|\nabla D\| \tag{20}$$

where $\sum_{D=0}$ means summation over all the dispersion curves and ds is a differential element of arc length of the dispersion curve. The function $\Sigma_1 = -\,\mathrm{sign}(\alpha)$ as already noted and the function Σ_2 is given by

$$\Sigma_2 = \mathrm{erf}(xD_\alpha + yD_\beta) \tag{21}$$

where $(D_\alpha, D_\beta) = (\partial D/\partial\alpha, \partial D/\partial\beta)$ and $\|\nabla D\| = \sqrt{D_\alpha^2 + D_\beta^2}$. The function $\mathrm{erf}\{\cdot\}$ in (21) is the usual error function defined in Abromowitz & Stegun (1967).

From (5), $D=0$ defines two distinct curves in the left-half $(\alpha<0)$ and right-half $(\alpha>0)$ planes symmetrical with respect to $\alpha=0$. Both dispersion curves are symmetrical with respect to $\beta=0$ as well. The dispersion curves $k=k(\theta)$ for $0\le\theta<\pi/2$ are depicted on Figure 1. The curves from right to left are associated with the value of $F/\sqrt{h} = (\mathbf{0}, 0.8, 0.9, 0.95, \mathbf{1}, 1.05, 1.1, 1.2, 2, 3, 5)$, respectively. It can be verified that the imaginary part of G^W given by (20) is nil due to the symmetrical properties of the dispersion curves, and the wave component G^W is depicted on Figures 2 and 3 for $F/\sqrt{h}=0.95$ (subcritical regime) and 2.0 (supercritical regime), respectively.

References

[1] Abramowitz, M. & Stegun, I.A. (1967) Handbook of mathematical functions. *Dover Publications.*

[2] Wahausen J.V. & Laitone E.V. (1960) Surface waves. *Encyclopedia of Phisics*, Pringer-Verlag.

[3] Noblesse F. & Chen X.B. (1995) Decomposition of free-surface effects into wave and near-field components. *Ship Tech. Res.* 42, 167-185.

[4] Chen X.B. & Nguyen T. (2000) Ship-motion Green function in finite-depth water. *4th Intl Conf. on HydroDynamics*, Yokohama (Japan).

Figure 1: Dispersion curves depending on the values of $F/\sqrt{h}$

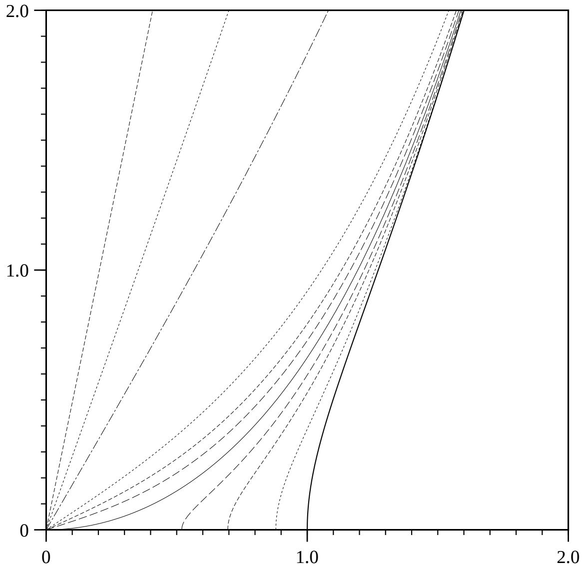

Figure 2: Wave component of the Green function at $F/\sqrt{h} = 0.95$ $(-10 \leq X \leq 2, -3 \leq Y \leq 3)$

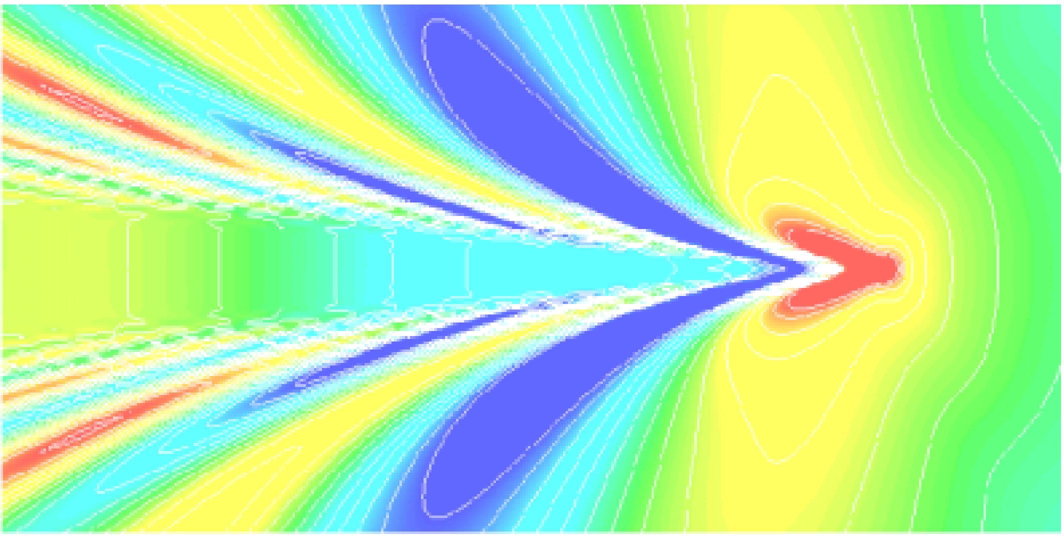

Figure 3: Wave component of the Green function at $F/\sqrt{h} = 2$ $(-10 \leq X \leq 2, -3 \leq Y \leq 3)$