WATER WAVE SCATTERING BY INCLINED BARRIER SUBMERGED IN FINITE DEPTH WATER

Mridula Kanoria[1] **and B.N. Mandal**[2]

[1] Department of Applied Mathematics, Calcutta University,India

[2] Physics and Applied Mathematics Unit, Indian Statistical Institute,India

1. Introduction : Water wave scattering problems involving thin barrier of arbitrary shape submerged in finite depth water are generally tackled by some approximate techniques assuming linear theory. The technique of hypersingular integral equation is demonstrated here in tackling the problem of water wave scattering by an inclined thin barrier submerged in *finite depth water*. The corresponding *deep water* problem was earlier investigated by Parson and Martin[1]. An appropriate use of Green's integral theorem produces a representation of the velocity potential describing the irrotational motion in the fluid region, in terms of the unknown dicontinuity of the potential across the submerged barrier. Utilization of the boundary condition on the barrier gives rise to an integro-differential equation for the discontinuity, which is interpreted as equivalent to a hypersingular integral equation. This is solved numerically by approximating the discontinuity in terms of a finite series involving Chebysev polynomials of the second kind followed by a collocation method. The reflection and transmission coefficients are then estimated numerically using this solution. The force and moment(about the origin) acting on the barrier per unit width are also estimated for various positions of the barrier. Comparison is made with available deep water results. It is observed that if the mid point of the barrier is submerged to the order of one-tenth of the bottom depth, then the deep water results effectively hold good, and in that case, the finite depth problem can be modelled as deep-water problem.

2. Mathematical formulation : Let a thin straight inclined barrier Γ be submerged in water of uniform finite depth h, and let d be the depth of its mid point below the mean free surface and the co-ordinate system be so chosen that the position of Γ is described by $y = d + ta\cos\theta, x = ta\sin\theta(-1 \leq t \leq 1, d > a\cos\theta, h > d + a\cos\theta, 0 \leq \theta \leq 90^0, \theta$ being the angle of inclination of the barrier with the vertical). A train of surface water waves of amplitude b_0 and circular frequency σ is incident from the direction of $x = -\infty$ on the barrier. The incident wave potential $Re\{\phi_0(x,y)e^{-i\sigma t}\}$ with

$$\phi_0(x,y) = \frac{gb_0}{\sigma}\frac{\cosh k_0(h-y)e^{ik_0x}}{\cosh k_0 h}$$

where k_0 is the unique positive real root of the transcendental equation $k\tanh kh = K$, with $K = \sigma^2/g$, g being the gravity. The ensuing motion in the fluid region described by velocity potential $Re\{\phi(x,y)e^{-i\sigma t}\}$ where $\phi(x,y)$ satisfies

$$\nabla^2\phi = 0, \quad 0 \leq y \leq h$$

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{on } y = 0,$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma$$

where $\frac{\partial}{\partial n}$ denotes normal derivatives on Γ,

$$r^{1/2}\nabla\phi \quad \text{is bounded as } r \to 0$$

where r is the distance from the submerged edges of Γ,

$$\frac{\partial \phi}{\partial y} = 0 \ \text{ on } y = h$$

$$\phi(x,y) \to \begin{cases} T\phi_0(x,y) & \text{as } x \to \infty, \\ \phi_0(x,y) + R\phi_0(-x,y) & \text{as } x \to -\infty \end{cases}$$

where R and T are respectively the unknown reflection and transmission coefficients (complex) which will be determined in the course of mathematical analysis.

3. The method of solution : By an appropriate use of Green's integral theorem in the fluid region a representation of $\phi(\xi,\eta)$ at a point $p \equiv (\xi,\eta)(0 < \eta < h)$ is found to be

$$\phi(\xi,\eta) = \phi_0(\xi,\eta) - \frac{1}{2\pi}\int_\Gamma F(q)\frac{\partial G}{\partial n_q}(x,y:\xi,\eta)ds_q \tag{3.1}$$

where $F(q)$ the discontinuity of $\phi(x,y)$ across Γ and $q \equiv (x,y)$ is a point on Γ, and $\frac{\partial}{\partial n_q}$ denotes the normal derivatives at q on Γ, and $G(p;q)$ is given by (cf Banerjea *et al* [2])

$$G(x,y:\xi,\eta) \;=\; ln\frac{r}{r'} - 2\int_{C_1}\frac{e^{-k(y+\eta)}}{k-K}\cos k(x-\xi)dk - 2\int_{C_2}\frac{e^{-kh}L(k,y)L(k,\eta)}{k(k-K)\Delta(k)}\cos k(x-\xi)dk \tag{3.2}$$

where $r, r' = \{(x-\xi)^2 + (y \mp \eta)^2\}^{1/2}$,

$$L(k,y) = k\cosh ky - K\sinh ky, \ \ \Delta(k) = k\sinh kh - K\cosh kh, \tag{3.3}$$

and the paths C_1, C_2 are along the positive real axis in the complex k-plane indented below the pole at $k = K$ for C_1 and below the poles at $k = K,\ k_0$ for C_2.The boundary condition on Γ produces an integro-differential equation, which can be interpreted as the following *hypersingular* integral equation in [1]

$$\frac{1}{2\pi}\,⨍_\Gamma F(q)\frac{\partial^2 G(p;q)}{\partial n_p \partial n_q}ds_q = \frac{\partial \phi_0}{\partial n_p}, \ \ p \in \Gamma \tag{3.4}$$

for the determination of $F(q)$, where the cross on the integral sign indicates that it is to be interpreted as a Hadamard finite part integral.

Denoting ξ, η by $\xi = ua\sin\theta, \ \ \eta = d + ua\cos\theta, \ \ -1 \le u \le 1$ it can be shown that

$$\frac{\partial^2 G(p;q)}{\partial n_p \partial n_q} = -\frac{1}{a^2}\left[\frac{1}{(u-t)^2} - \mathcal{K}(u,t)\right]. \tag{3.5}$$

where $\mathcal{K}(u,t)$ is a regular function of u, t, and can be expanded in a form suitable for numerical computation. The details of this expansion is ommitted here. However, this expansion is an important step in this work.

The hypersingular integral equation (3.4) can be rewritten as

$$⨍_{-1}^{1}\left[\frac{1}{(u-t)^2} + \mathcal{K}(u,t)\right]f(t)dt = l(u), \quad -1 < u < 1 \tag{3.6}$$

$$f(t) = \frac{gb_0}{\sigma}F(t) \tag{3.7}$$

and $l(u)$ is a known function (it is related to ϕ_0). The equation (3.6) is to be solved subject to the condition that

$$f(\pm 1) = 0. \tag{3.8}$$

As in [1] $f(t)$ is approximated as

$$f(t) = (1 - t^2)^{1/2} \sum_{n=0}^{N} a_n U_n(t) \tag{3.9}$$

where $U_n(t)$ is Chebyshev polynomial of the second kind. and $a_n(n = 0, 1, ..N)$ are unknown complex constants. Substituting (3.9) into (3.6), we obtain

$$\sum_{n=0}^{N} a_n A_n(u) = l(u), \quad -1 < u < 1, \tag{3.10}$$

where $A_n(u) = -\pi(n+1)U_n(u) + \int_{-1}^{1} (1 - t^2)^{1/2} \mathcal{K}(u,t) U_n(t) dt.$

To find the unknown constant $a_n(n = 0, 1, ...N)$ we put $u = u_j(j = 0, 1, ..N)$ in the relation (3.10) to obtain the linear system

$$\sum_{n=0}^{N} a_n A_n(u_j) = l(u_j) \quad j = 0, 1, ...N \tag{3.11}$$

The collocation points $u_j(j = 0, 1, ..N)$ are chosen as [1]

The reflection and transmission coefficients R and T can be found in terms of a series involving a_n by making $\xi \to \mp\infty$ in the expression for $\phi(\xi, \eta)$ given in (3.1). Expressions for R, T involve certain integrals which can be evaluated numerically for different values of the physical parameters $Ka, h/a, d/a$ and the angle θ of inclination of the barrier with the vertical. Thus once the complex constants $a_n(n = 0, 1, ..N)$ are found by solving the linear system (3.11), numerical estimates for $|R|$ and $|T|$ can be obtained. Also $|R|^2 + |T|^2$ must be unity from the consideration of energy principle, this can be used to check the correctness of numerical estimates obtained for $|R|$ and $|T|$. The amplitudes of the force and moment (about the origin) per unit width of the barrier can also be estimated numerically once $a_n(n = 0, 2, ..N)$ are obtained.

4. Numerical results : The numerical estimates for $|R|$ converges fairly rapidly with N. An accuracy of almost five decimal places has been achieved by choosing $N = 3$ or 4. Also the correctness of the numerical method is checked by estimating $|R|$ for the case of a submerged vertical plate ($\theta = 0^0$) and comparing with known results obtained earlier in [3] by eignefunction expansion method. Again, for $\theta = 45^0, a/h = 0.4, d/h = 0.4$, $|R|$ and $|T|$ are estimated by the present method, and it has been verified that $|R|^2 + |T|^2$ almost coincide with unity.

$|R|$ is depicted against the wave number Ka in figure 1 taking $\theta = 45^0$ for different values of d/h keeping d/a fixed. It is observed that when the depth of the mid point of the inclined barrier is one-tenth of the bottom depth ($d/h = 0.1$), the results almost coincide with deep-water results given in [1] and shown in the same figure by crosses. The bottom effect appears to be significant in the low wave number range. This may be attributed due to the fact that in the low wave number range, the wave length of the incident wave train is large enough to have adequate penetration below the free surface so as to be affected significantly by the bottom while in the large wave number range the reverse phenomenon occurs and as such there is no appreciable effect is observed.

In figure 2 $|R|$ depicted against Ka for an almost horizontal barrier by taking $\theta = 89^0$. As observed in [1] for the case of deep water, zero of $|R|$ are seen to occur for the case of finite depth water. The zeros of $|R|$ begin to appear only when the inclination of the barrier with the vertical is of the order of 80^0. This is not shown here. The non-dimensional amplitude of the force and moment(about the origin) acting on the barrier per unit width are depicted in figure 3 and 4 respectively. It is observed from these figures that the amplitudes decrease with the increase of θ. These results are plausible.

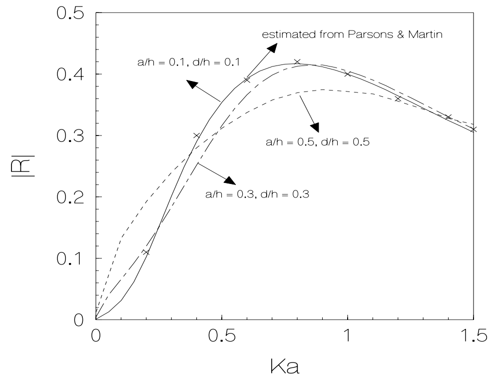

Fig. 1: Reflection coefficient vs wave number, $\theta = 45^0$

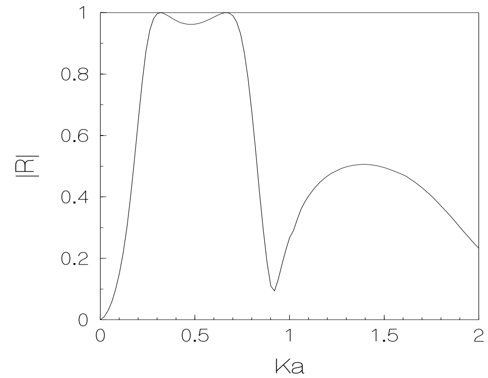

Fig. 2: Reflection coefficient vs wave number, a/h = 0.8,d/h = 0.1,$\theta = 89^0$

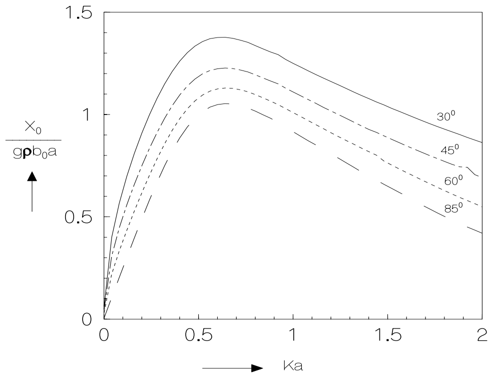

Fig. 3: Amplitude of the force per unit width vs wave number
a/h=.3,d/h=.3

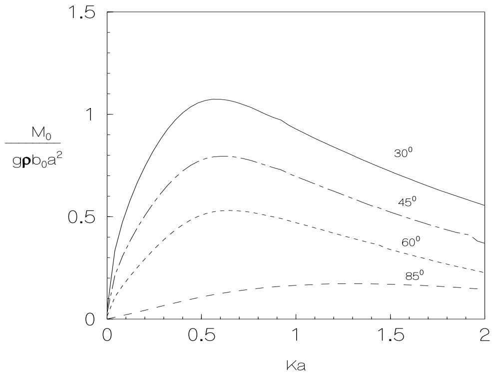

Fig. 4: Amplitude of moment per unit width vs wave number
d/h=0.3,a/h=0.3

5. Conclusion : Water wave scattering by an inclined barrier plate submerged in *finite depth* water is investigated by using a hypersingular integral equation formulation. Numerical estimates for the reflection coefficient, amplitudes of force and moment acting on the barrier are obtained fairly accurately and are depicted graphically against the wave number and compared with deep-water results. Also, some numerical results for an almost horizontal barrier are obtained as special cases, and these agree with known results obtained by other methods. The technique used here is now being used to tackle water wave scattering problems involving a *curved* barrier in the form of an arc of a circle, ellipse etc. submerged in finite depth water . The method can also be used with some modification for a surface piercing curved barrier.

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References

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