Resonances for cylinder arrays

by C M Linton & M McIver
Department of Mathematical Sciences, Loughborough University,
Leicestershire, LE11 3TU, UK

INTRODUCTION

The importance of trapped modes in the design of offshore structures which are supported by large arrays of vertical cylindrical structures was brought to prominence by Maniar and Newman (1997) who, in an investigation into the scattering of surface waves by a long but finite array of bottom-mounted vertical circular cylinders, discovered that at particular frequencies the hydrodynamic loads on the cylinders could become abnormally large. They identified this phenomenon with the existence of resonant trapped modes when cylinders are placed in channels on the walls of which either Neumann or Dirichlet boundary conditions are applied. For the Neumann case such modes were well known, but Maniar and Newman's observation that such modes could exist when the potential rather than its normal derivative was made to vanish on the walls, provided the cylinder radius was smaller than some critical value, was new. The trapped modes that can exist when N circular cylinders are placed across a channel in such a way that they form a section of an infinite array of equally-spaced cylinders were subsequently investigated by Utsunomiya and Eatock Taylor (1999). By representing the solution as a series of multipole potentials they were able to numerically compute N distinct trapped modes for any given cylinder radius when Neumann boundary conditions were applied on the tank walls, but for the case of Dirichlet boundary conditions they found $N-1$ or N modes depending on whether the cylinder radius was greater than or less than some critical value.

Porter and Evans (1999) used an integral equation technique to investigate the more general phenomenon of Rayleigh-Bloch surface waves (for which no general existence criteria are known) travelling along arbitrary periodic structures. Such waves are characterized by two parameters, k and β, the first being related to the frequency and the second corresponding to the dominant wavenumber in the direction along the structure. The parameter β provides a natural cut-off in that for values of k less than β energy cannot propagate away from the structure and so it is possible to look for specific values of $k < \beta$ at which pure Rayleigh-Bloch surface waves can occur. Porter and Evans showed that for certain discrete values of β these modes may correspond to trapped modes in the vicinity of a finite array of cylinders spanning a channel; precisely the situation studied by Utsunomiya and Eatock Taylor for an array of circular cylinders. If one assumes the existence of pure Rayleigh-Bloch surface waves for a particular periodic structure, then Porter and Evans' work explains the results of Utsunomiya and Eatock Taylor.

The purpose of this paper is to show that the channel modes found by Utsunomiya and Eatock Taylor and by Porter and Evans correspond to discrete eigenvalues below the continuous spectrum for certain differential operators and to show how standard variational arguments can be used to prove their existence. The key ingredient is a decomposition theorem which shows that functions $f(y)$ defined on domains which are both periodic and symmetric about zero and which also satisfy conditions equivalent to Neumann or Dirichlet boundary conditions on $y = 0$ and $2N$ can be decomposed into $N + 1$ orthogonal functions. This is a direct extension of the decomposition of a function defined on a symmetric domain into its symmetric and antisymmetric parts (which corresponds to the case $N = 1$). This result can be applied to the scattering potentials for channels containing periodic structures and the class of all such potentials decomposed into $N+1$ subclasses. Green's theorem then shows that an incident wave in a particular class only scatters waves from the same class.

Spectral theory can be used to show that $N + 1$ operators exist for each problem, all of whose continuous spectra are bounded away from zero in the Dirichlet case and N of which have this property in the Neumann case. The existence of trapped modes then follows from a standard variational argument.

DECOMPOSITION THEOREM

Assume that $D \subset \mathbb{R}$ is periodic with period 2 and also symmetric about zero, i.e. if $y \in D$ then $y+2 \in D$ and $-y \in D$ (from which it follows that $2n \pm y \in D$, $n \in \mathbb{Z}$). Let $f : D \cap [0, 2N] \to \mathbb{R}$, $N \in \mathbb{N}$, be given and extend it to a function on the whole of D by either

$$f(-y) = f(y), \qquad f(2N+y) = f(2N-y), \tag{1}$$

which will be referred to as the Neumann case, or

$$f(-y) = -f(y), \qquad f(2N+y) = -f(2N-y), \tag{2}$$

which will be referred to as the Dirichlet case. Under these conditions

$$f(y) = \sum_{m=0}^{N} f_m(y), \tag{3}$$

where

$$f_m(y) = \frac{\gamma_m^N}{2N} \sum_{n=1-N}^{N} c_{mn}^N f(y+2n), \tag{4}$$

$$c_{mn}^N = \cos\frac{mn\pi}{N}, \qquad \gamma_m^N = \frac{2}{1+\delta_{m0}+\delta_{mN}} \tag{5}$$

and δ_{mn} is the Kronecker delta.

Furthermore, suppose that two functions f and g, which are defined on the same domain D and which both satisfy either (1) or (2), are decomposed according to (3) and (4), then the following orthogonality result holds:

$$\int_0^{2N} f_m(y) g_\mu(y)\,\mathrm{d}y = \delta_{m\mu} \frac{\gamma_m^N}{2N} \sum_{s=-N}^{N-1} \sum_{\sigma=-N}^{N-1} c_{m,s-\sigma}^N \int_0^1 f(y+2s) g(y+2\sigma)\,\mathrm{d}y. \tag{6}$$

When $N = 1$ this decomposition theorem is nothing more than the splitting of a function defined on a symmetric domain into its symmetric and antisymmetric parts. For $N > 2$ the symmetry properties of the functions f_m are still of interest, but they are insufficient to completely characterize the decomposition.

As an example, consider the Green's function for the two-dimensional Helmholtz equation $(\nabla^2 + k^2)\phi = 0$ in a channel of width $2N$, satisfying Neumann boundary conditions on the guide walls. One way to represent this function is as an eigenfunction expansion and then from (3) we can obtain

$$G(x-\xi, y, \eta) = \sum_{m=0}^{N} G_m(x-\xi, y, \eta), \tag{7}$$

where

$$G_m(x-\xi, y, \eta) = -\frac{\gamma_m^N}{4N} \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\alpha_{mn}|x-\xi|}}{\alpha_{mn}} \cos\beta_{mn} y \cos\beta_{mn}\eta \tag{8}$$

and

$$\alpha_{mn} = \left(\beta_{mn}^2 - k^2\right)^{1/2} = -\mathrm{i}\left(k^2 - \beta_{mn}^2\right)^{1/2}, \quad \beta_{mn} = \frac{(m+2nN)\pi}{2N}. \tag{9}$$

The function G_m represents a sum of unequal sources at $x = \xi$, $y = 2n \pm \eta$, $n \in \mathbb{Z}$ and thus has $2N$ singularities within the guide (at $y = \eta$, $y = 2n \pm \eta$, $n = 1, \ldots, N-1$, $y = 2N - \eta$) even though the combination of these functions given by (7) only has one such singularity.

Suppose now that we wish to solve a scattering problem in a channel spanned by an array of cylinders. The potential ϕ can be split up into $N+1$ orthogonal functions and if we apply Green's theorem to ϕ and G_m we obtain

$$\frac{1}{2}\phi_m(\xi, \eta) = \int_B \phi_m \frac{\partial G_m}{\partial n}\,\mathrm{d}s + \chi_m, \qquad (\xi, \eta) \in B, \tag{10}$$

where χ_m comes from the decomposition of the incident wave and B is the boundary of the cylinder array. Thus, for the special types of geometry under consideration, an arbitrary scattering problem can be decomposed into $N+1$ independent problems.

This result is of both practical and theoretical importance. From a practical point of view, the decomposition leads to a significant computational saving when calculating the effects of scattering by an array of cylinders spanning a channel, and from a theoretical viewpoint it enables us to prove the existence of trapped modes as described below.

RESONANCES

If we denote the fluid domain by Ω, then the solution to a scattering problem of the type described above will not be unique if a non-trivial solution to the homogeneous boundary-value problem

$$(\nabla^2 + k^2)\phi = 0 \quad \text{in} \quad \Omega, \tag{11}$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = 0, 2N, \tag{12}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad B, \tag{13}$$

$$\phi \to 0 \quad \text{as} \quad |x| \to \infty, \tag{14}$$

exists. If such a solution exists for a given k^2 then k^2 is an eigenvalue of $-\nabla^2$ (with Neumann boundary conditions) on the domain Ω. The solution itself (which is known as a trapped mode or an acoustic resonance) is the corresponding eigenvector.

In terms of the spectral theory of operators we can think of (11)–(14) as being an eigenvalue problem for an operator A consisting of $-\nabla^2$ in Ω together with the various boundary conditions. The spectrum of A is a closed set containing all the values of k^2 for which the operator $A - k^2 I$ does not have a bounded inverse and for the above problem this set can be divided into two disjoint subsets. The values of k^2 for which the operator $A - k^2 I$ is not invertible are called eigenvalues of A and together they make up the point spectrum of A, denoted by $\sigma_p(A)$. It is well-known that for the problem specified in (11)–(14) any eigenvalues are real and non-negative. The remainder of the spectrum of A is called the continuous spectrum and denoted by $\sigma_c(A)$. We can also consider the set of values of k^2 for which we can set up a wave scattering problem. This set is the essential spectrum of A and is denoted by $\sigma_{\text{ess}}(A)$. The essential spectrum is the union of the continuous spectrum and any embedded eigenvalues.

It is well-known that $\sigma_{\text{ess}}(A) = [0, \infty)$ and so any eigenvalue of A is necessarily embedded in the continuous spectrum which makes analysis of these eigenvalues difficult. One way of overcoming the problem is to find a decomposition of the space of functions on which A operates, $S = S_0 \oplus S_1$ say ($\oplus$ denotes direct sum), such that when we consider the operator A restricted to one of these subspaces the continuous spectrum is moved away from the origin. An example is provided by Evans, Levitin, and Vassiliev (1994) who considered the case $N = 1$. Below we extend this decomposition to arbitrary $N \in \mathbb{N}$ which allows N eigenvalues to be found for each geometry.

Suppose then that we have a channel spanned by a periodic structure. The decomposition theorem shows that any function which is defined on Ω can be written as

$$\phi(x, y) = \sum_{m=0}^{N} \phi_m(x, y) \tag{15}$$

and we can thus decompose the space of square integrable functions in the fluid region Ω as

$$L^2(\Omega) = S_0 \oplus S_1 \oplus \cdots \oplus S_N, \tag{16}$$

where S_m is the space of functions of the form $\phi_m(x, y)$. It makes sense to restrict the operator A to one of the spaces S_m and this restricted operator will be labelled A_m. Within a given space

S_m, waves can only exist if $k^2 > m^2\pi^2/4N^2$. We write $\beta_m = m\pi/2N$ and then it follows that the essential spectrum for A_m is given by

$$\sigma_{\text{ess}}(A_m) = [\beta_m^2, \infty) \tag{17}$$

and so there is a non-zero cut-off for each of the spaces except for S_0. Standard variational arguments can then be used to show that the operator A_m, $m \in \{1, \ldots, N\}$ has at least one eigenvalue less than β_m^2.

Different results apply to the Dirichlet case. Thus we consider the homogeneous boundary-value problem (11)–(14) with (12) replaced by

$$\phi = 0 \quad \text{on} \quad y = 0, 2N, \tag{18}$$

though we will still refer to the associated operator as A. The space $L^2(\Omega)$ can be decomposed exactly as before. The essential spectrum for A_m is now given by

$$\sigma_{\text{ess}}(A_m) = \begin{cases} [\beta_m^2, \infty) & m = 1, \ldots, N, \\ [\pi^2, \infty) & m = 0 \end{cases} \tag{19}$$

and so there is a non-zero cut-off for all of the spaces. However, the case $m = N$ corresponds to a single body symmetrically placed about the centreline of a channel of width 2 with Dirichlet conditions on the walls and it has been proven (see McIver and Linton 1995, p548) that no trapped modes can exist in this case for $k < \pi/2 = \beta_N$. Hence we can only hope to prove the existence of trapped modes for $m = 0, \ldots, N-1$.

Variational arguments in fact show that the operator A_m, $m \in \{1, \ldots, N-1\}$ has at least one eigenvalue and that the operator A_0 has at least one eigenvalue if the body shape satisfies some geometric condition.

References

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