FORCE ON A BODY IN A UNIFORMLY STRATIFIED FLUID: AFFINE SIMILITUDE

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1 INTRODUCTION

Theoretical description of low-frequency oscillations (at the time-scale of minutes) of marine structures and deep-submersibles in a real stratified sea environment should take into account the internal-wave radiation. Real smooth density profiles observed in nature are usually idealized theoretically either as a two-fluid system with an interface or as a uniformly stratified fluid of infinite extent. In the former case, one can apply the methods developed in the theory of surface waves and consider the piecewise-potential flow with appropriate conditions at the interface. In the latter case, the problem bears similarities with the theory of a thin wing in a flow of compressible fluid. Indeed, the fundamental parameter of the problem $\Omega = \omega/N$ (the ratio between the frequency of the body oscillations ω and the buoyancy frequency N) physically plays the role of the Froude number, while being formally analogous to the inverse Mach number. In the thin-wing theory, the notion of affine similitude is used for derivation of the Prandtl-Glauert formula, which relates the lift forces on the affinely-similar wing profiles in the subsonic flow of compressible and incompressible fluids.

In the present paper we exploite a similar idea and obtain a simple formula, which gives a relation between the tensors of the added mass coefficients of affinely-similar bodies oscillating in homogeneous and uniformly stratified ideal fluids at $\Omega > 1$ (elliptic problem). Since the problem on the oscillations of a body in a uniformly stratified fluid can, in principle, be formulated in time-domain in terms of the causal Green function, the solution of hyperbolic problem ($\Omega < 1$) in frequency-domain can be obtained by analytic continuation. The known solutions [1 - 4] for hydrodynamic loads acting on the bodies of particular geometry can be obtained from the relations discussed in the present paper. The rule of affine similitude formulated in the present paper is confirmed by experiments with spheroids having different length-to-diameter ratios. Furthermore, we focuss our attention on the oscillations of horizontal cylinders with the polygon shape of cross-sections (rhomb and square) and demonstrate the existence of specific critical regimes that occur at certain Ω_*, when the slope of the characteristic lines of the governing hyperbolic equation coincide with the slope of the sides of the polygon profiles. The experimental technique for evaluation of the frequency-dependent force coefficients is based on Fourier-analysis of the time-history of the damped oscillation tests [5].

2 THEORETICAL ANALYSIS

Let us consider harmonical oscillations of a body with frequency ω in an inviscid uniformly stratified Boussinesq fluid with constant Brunt-Vaisala frequency $N(x_3) = [(-g/\rho)d\rho/dx_3]^{1/2}$, where $\rho(x_3)$ is the density distribution over vertical coordinate of the Cartesian coordinate system (x_1, x_2, x_3), and g is the gravity acceleration. Assuming that the body velocity $\mathbf{v}^{(\mathbf{1})}$, 'internal' potential $\phi^{(1)}$, fluid velocity $\mathbf{u}^{(\mathbf{1})}$ and pressure $p^{(1)}$ can be all represented in the form $\mathbf{v}^{(1)}(x_1, x_2, x_3, t) = \mathbf{V}^{(1)}(x_1, x_2, x_3)\exp(\mathrm{i}\omega t)$, etc., the equation of fluid motion in terms of 'internal' potential [6] becomes

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{\alpha^2}\frac{\partial^2}{\partial x_3^2}\right)\Phi^{(1)} = 0, \tag{1}$$

where $\alpha = (\Omega^2 - 1)^{1/2}/\Omega$, with $\Omega = \omega/N$. Hereinafter, superscript (1) is assigned to the variables of Problem 1. Equation (1) may be of elliptic or hyperbolic type depending on the sign of α^2. First, we consider elliptic problem ($\alpha^2 > 0$). The fluid velocity in Problem 1 is given by

$$\mathbf{U}^{(1)} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{1}{\alpha^2}\frac{\partial}{\partial x_3}\right)\Phi^{(1)}$$

The impermeability condition on the body surface $S^{(1)}$ given by $F^{(1)}(x_1, x_2, x_3) = 0$ is

$$\mathbf{U}^{(1)} \cdot \nabla^{(1)} F^{(1)} = \mathbf{V}^{(1)} \cdot \nabla^{(1)} F^{(1)},$$

where $\nabla^{(1)} = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. On affine transformation of the coordinate system

$$\zeta_i = a_i x_i, \qquad a_i = (1, 1, \alpha), \tag{2}$$

the governing equation (1) in Problem 1 becomes the Laplace equation in Problem 2 (hereinafter, superscript (2) denotes the variables of Problem 2)

$$\left(\frac{\partial^2}{\partial \zeta_1^2} + \frac{\partial^2}{\partial \zeta_2^2} + \frac{\partial^2}{\partial \zeta_3^2}\right)\Phi^{(2)} = 0. \tag{3}$$

The impermeability condition on the body surface $S^{(2)}$ described by the function $F^{(2)}(\zeta_1, \zeta_2, \zeta_3) = 0$ takes the form

$$\nabla^{(2)}\Phi^{(2)} \cdot \nabla^{(2)} F^{(2)} = \mathbf{V}^{(2)} \cdot \nabla^{(2)} F^{(2)} \tag{4}$$

where $\nabla^{(2)} = (\partial/\partial\zeta_1, \partial/\partial\zeta_2, \partial/\partial\zeta_3)$, and the components of vector of the body velocity are $V_i^{(2)} = a_i V_i^{(1)}$.The components of the vector of hydrodynamic load $\mathbf{y}^{(1,2)}(t) = \mathbf{Y}^{(1,2)}\exp(i\omega t)$ in Problems 1 and 2 can be found by integrating pressure $P^{(1,2)} = -\rho\mathrm{i}\omega\Phi^{(1,2)}$ over the body surface $S^{(1,2)}$.

As evident from equations (3) and (4), Problem 2 is the classic problem on oscillations of a body in a homogeneous ideal fluid. It is well-known [8] that the components of hydrodynamic loads in Problem 2 can be expressed in terms of the added mass tensor. Similar concept can be introduced for Problem 1. It can be shown [7] that the tensors of the added mass coefficients in Problems 1 and 2 are related as follows

$$K_{ij}^{(1)} = K_{ij}^{(2)} a_i a_j, \tag{5}$$

with $K_{ij}^{(1)} = m_{ij}^{(1)}/\rho W^{(1)}$ and $K_{ij}^{(2)} = m_{ij}^{(2)}/\rho W^{(2)}$, where $W^{(1)}$ and $W^{(2)}$ are the volumes of bodies surrounded by $S^{(1)}$ and $S^{(2)}$, respectively. Now let us suppose that for a certain family of bodies oscillating in ideal homogeneous fluid we know the functions, representing the dependence of the added mass coefficients on non-dimensional geometrical parameters $K_{ij}^{(2)} = f_{ij}(e, q)$, where $e = b_2/b_1$ and $q = b_3/b_1$ are the relations between the characteristic dimensions b_1, b_2, b_3 of the bodies along the directions $\zeta_1, \zeta_2, \zeta_3$. Then, the solution of Problem 1 for a body with given e_0 and q_0 can be found from (5) as follows

$$K_{ij}^{(1)}(\Omega) = f_{ij}(e_0, q_0\alpha) a_i a_j. \tag{6}$$

At $\Omega < 0$ equation (1) is of hyperbolic type ($\alpha^2 < 0$). The characteristic feature of the hyperbolic problem is radiation of internal waves by the oscillating body. The solution of the hyperbolic problem can be obtained by analytic continuation in frequency. The analytic continuation for α is $-\mathrm{i}\eta$, where $\eta = (1 - \Omega^2)^{1/2}/\Omega$ is the real-value parameter. The coefficients a_i introduced in (2) should be replaced by $\gamma_i = (1, 1, -\mathrm{i}\eta)$. Accordingly, expression (6) becomes

$$K_{ij}^{(1)}(\Omega) = f_{ij}(e_0, -q_0\mathrm{i}\eta)\gamma_i\gamma_j.$$

The functions $f_{ij}(e, q)$ for different bodies can be found in appropriate handbooks (see, for example, [9]). In particular, the functions $f_{ij}(e, q)$ for spheroids and elliptic cylinders yield the solutions [1 - 4], obtained by different approaches. Note that in 2D case the functions f_{ij} are the functions of a single argument q, what raises some important consequences for low-frequency asymptotic of the force coefficients and for application of the Kramers-Kronig relations [10] relating the real and imaginary parts of $K_{ij}^{(1)}(\Omega)$. We use the standard decomposition of the complex force coefficients as is customary in naval hydrodynamics [8], i.e. $m_{ij}^{(1)} = \mu_{ij} - \mathrm{i}\lambda_{ij}/\omega$, where μ_{ij} and λ_{ij} are the added masses and damping coefficients, respectively. The corresponding non-dimensional values are introduced as $C_{ij}^{\mu} = \Re[K_{ij}^{(1)}] = \mu_{ij}/\rho W^{(1)}$ and $C_{ij}^{\lambda} = \Omega\Im[K_{ij}^{(1)}] = \lambda_{ij}/\rho N W^{(1)}$. In 2D case, the coefficients are defined for cylinders of unit length and $W^{(1)}$ is the cross-sectional area.

For a set of affinely-similar bodies with different q_0 and fixed e_0, we can formulate the rule of affine similitude that requires

$$K_{ij}^{(1)}/(a_i a_j) = idem \qquad \text{if} \qquad q_0^2(\Omega^2 - 1)/\Omega^2 = q_0^2\alpha^2 = idem.$$

3 EXPERIMENTS

The experimental technique used in the present study is essentially similar to the one described in [5]. The frequency-dependent force coefficients $C_{11}^{\mu}(\Omega)$ and $C_{11}^{\lambda}(\Omega)$ for a horizontally oscillating body are evaluated from Fourier-transforms of the impulse response function of the body attached to the lower end of a pendulum having variable restoring moment. Experiments were performed with 2D (cylinders with different shapes of cross-section) and 3D (spheroids with different q_0) bodies. The theoretical and experimental results are exemplified in figure 1*a*, *b* representing the added mass and damping coefficients of rhomb and square cylinders oscillating in a uniformly stratified fluid. For the data obtained in homogeneous fluid (line *4* in figure 1*b*) the normalization of λ_{11} by the value $\rho_0 N W^{(1)}$, which took place in the stratified-fluid experiments, has purely formal sense. The limiting value $C_{11}^{\mu}(\infty) \approx 1.19$ is common both for the rhomb and square profiles. Here, the rhomb is a square with vertical diagonal. Note that for $\Omega_* = 1/\sqrt{2}$ the force coefficients of the rhomb profile undergo a drastic change. At the frequency Ω_* the slope of the characteristic lines of the governing hyperbolic equation (that is equal to the slope of the group-velocity vector of internal waves) coincides with the slope of the sides of the rhomb.

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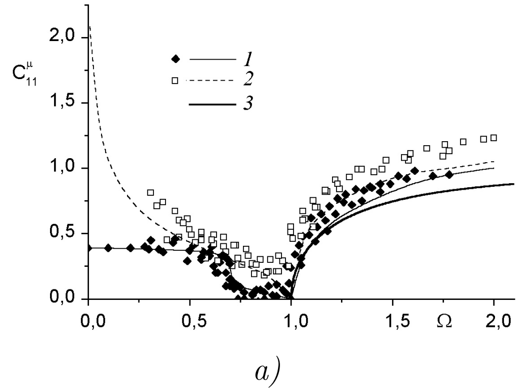

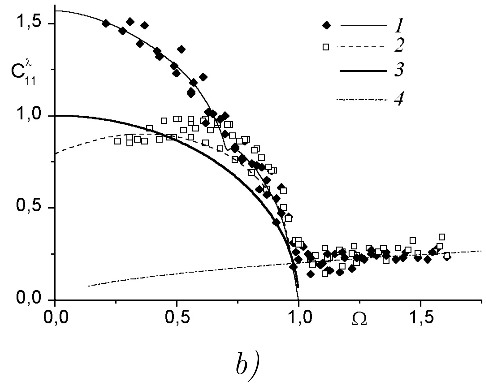

Figure 1: Added mass coefficient C_{11}^{μ} *a)* and damping coefficient C_{11}^{λ} *b)* versus frequency Ω (lines *1*, *2* - theory, symbols *1*, *2* - experimental points for rhomb and square, respectively, line *3* - theory [2] for a circular cylinder, line *4* - approximation of experimental data for the damping coefficients of the rhomb and square cylinders in homogeneous fluid)

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